LESSON 7

ANALYTICAL GEOMETRY

Analytical geometry in Gr12 mostly involves circles and tangents to circles. You will however need all the skills learnt in Gr11 to answer the questions.

Equations of circles.

The general equation for a circle with centre at the origin and radius r is given by $x^2 + y^2 = r^2$.

Consider the following:

Example

Example 1

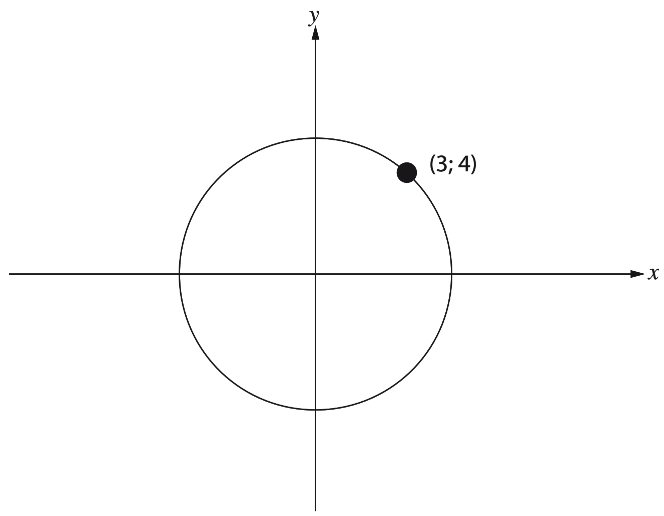
$x^2 + y^2 = 16$

this equation tells us that the centre of the circle is (0 ; 0) and the radius is 4 units (as $r^2 = 16 \therefore\ r = 14$)

Example

Example 2

Have a look at the diagram below



It is given that the centre of the circle is the origin and and that the circle passes through (3 ; 4).

Let's have a look at how to find the equation of the circle.

We start by using the general equation $x^2 + y^2 = r^2$ then we substitute our point (3 ; 4) into the equation

$3^2 + 4^2 = r^2$

$r^2 = 25$

Solving for r and substituting into the equation gives $x^2 + y^2 = 25$.

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Activity

Activity 1

Find the equation of the following circles with centre at the origin and:

1. radius $\sqrt{3}$ units

2. passing through the point (–5 ; 12)

3. passing through $\left(\frac{1}{2};\frac{1}{2}\right)$

Now let's have a look at circles with centre not at the origin.

These circles have the general equation $(x - a)^2 + (y - b)^2 = r^2$ where a and b are the x and y co-ordinates of the centre.

Example

Example 1

$(x - 2)^2 + (y + 3)^2 = 25$

The centre of the circle is (2 ; –3) and the radius of the circle is 5 units.

If the equation of a circle is not given in the form $(x - a)^2 + (y - \text{b})^2 = r^2$ we need to be able to complete the square in order to find the co-ordinates of the centre of the circle as well as the length of the radius.

Let's have a look at an example

Example

Example 2

Given a circle with equation: $x^2 + y^2 - 6x + 2y + 8 = 0$

Rewrite in the form $(x - a)^2 + (y - b)^2 = r^2$ and give the co-ordinates of the centre of the circle and the radius.

Step 1 Rewrite the equation $x^2 - 6x\ldots + y^2 + 2y\ldots = -8$ The x and y terms are written seperately and the constant term is moved to the right hand side of the equation.

Step 2 Halve the co-efficient of x and add the square of the result on both sides of the equation. Repeat the same process for y.
$x^2 - 6x + (-3)^2 + y^2 + 2y + (1)^2 = -8 + 9 + 1$

Step 3 Factorise $(x - 3)^2 + (y + 1)^2 = 2$

$\therefore$ co-ordinates of the centre of the circle (3 ; –1) and he radius is $\sqrt{2}$ units.

This is an important skill as it is used in most questions involving circles.

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Do Activity 2 to ensure that you are comfortable with the steps.

Activity

Activity 2

Determine the co-ordinates of the centre of the circle and the radius for each of the following:

1. $x^2 + y^2 - x - 2y - 5 = 0$

....

....

....

2. $x^2 + y^2 + 2x - 6y + 9 = 0$

....

....

....

3. $x^2 + y^2 - 4x - 6y + 9 = 0$

....

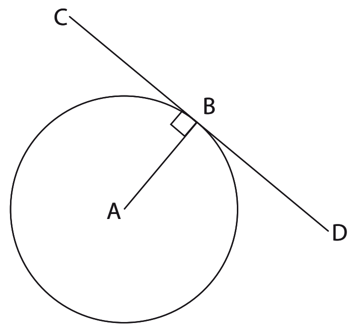
....

....

Equation of a tangent to a circle.

It is important to be able to find the equation of a tangent to a circle.

In the diagram alongside CBD is a tangent to the circle with centre A.



- A tangent is a straight line in the form $y = mx + c$.
- In order to find the equation of a tangent it is important to know that:

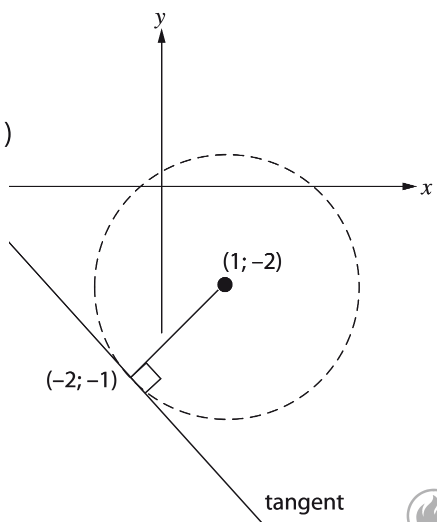
  $m_{radius} \times m_{tangent} = -1$ this means the radius and the tangent form of a 90° angle at the point of contact of the tangent.

Let's go through the steps of finding the equation of tangent.

Example

Example 1

Determine the equation of the tangent to the circle $(x - 1)^2 + (y + 2)^2 = 10$ at the point (–2 ; –1)



Step 1. Write down the co-ordinates of the centre of the circle and draw a rough diagram; Centre (1 ; –2)

Step 2 . Calculate the gradient of the radius.

$$m_{radius} = \frac{y_2 - y_1}{x_2 - x_2}$$

$$= \frac{-2-(-1)}{1-(-2)}$$

$$= \frac{-1}{3}$$

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Step 3. Determine the gradient of the tangent using $m_{radius} \times m_{tangent} = -1$

$m_{radius} = -\frac{1}{3}$

$\therefore m_{tangent} = 3$

Step 4. Use $y - y_1 = m(x - x_1)$ and the point of contact to find the equation of the tangent.

$y - (-1) = 3(x - (-2))$

$y + 1 = 3x + 6$

$y = x + 5$ (equation of the tangent)

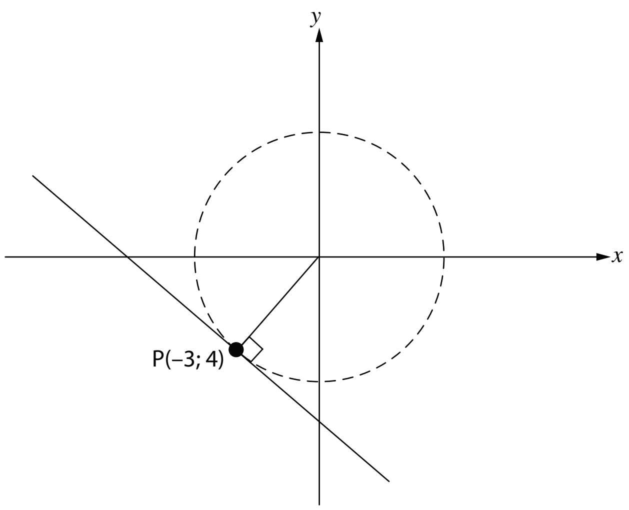
Let's work through another example.

Example

Example 2

Given the circle $x^2 + y^2 = 13$ Determine the equation of the tangent to the circle at P(–3 ; –2)

Step 1. Write down the co-ordinates of the centre of the circle and draw a rough diagram. As the equation of the circle is in the form $x^2 + y^2 = r^2$. We know that the centre of the circle is the origin.



Step 2. Calculate the gradient of the radius.

$m_r = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{0 - (-3)}{0 - (-2)}$

$= \frac{3}{2}$

Step 3. Use $m_{radius} \times m_{tangent} = -1$ to find the gradient of the tangent $m_{tangent} = -\frac{2}{3}$

Step 4. Use $y - y_1 = m(x - x_1)$ to solve

$y - (-2) = -\frac{2}{3}(x - (-3))$

$y + 2 = -\frac{2}{3} = (x + 3) \times 3$

$3y + 6 = -2x - 6$

$3y = 2x - 12$

$y = \frac{2}{3}x - 4$

Let's now have a look at some other questions involving tangents.

Example

Example 3

The equation of a circle is given by $x^2 + y^2 = 10$ Prove that $y = 3x + 10$ is a tangent to the circle.

We need to solve the simultaneous equations.

$y = 3x + 10\ldots$ (1)

$x^2 + y^2 = 10\ldots$ (2)

Sub (1) into (2)

$x^2 + (3x + 10)^2 = 10$

$x^2 + 9x^2 + 60x + 100 = 10$

$10x^2 + 60x + 90 = 0$

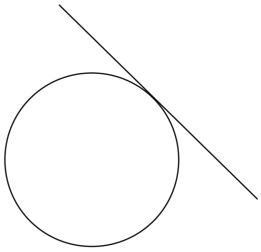
$x^2 + 6x + 9 = 0$

$(x + 3)(x + 3) = 0$

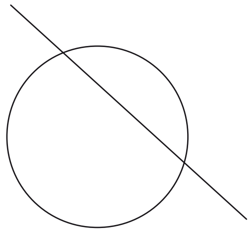
$x = 3$

What have we just proven? The fact there is only one solution if the equations are solved simultaneously. This shows that the line $y = 3x + 10$ just touches the circle in one place and therefore it is a tangent.

If we have 2 solutions it proves that the line intersects the circle of 2 places and is therefore not a tangent.

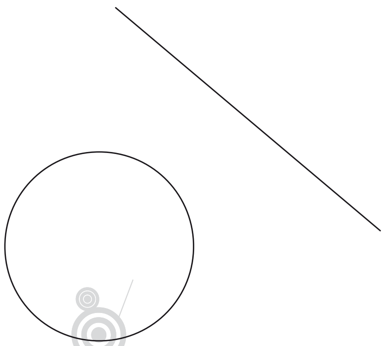


Tangent (only 1 solution)



(2 solutions)

If the simultaneous yields no solution then the line and the circle do not intersect



line and circle do not intersect.
(no solution)

Length of a tangent to a circle.

Let's have a look at how we use the skills we have learnt so far to calculate the length of a tangent.

Example

Example 1

Determine the length of the section of the tangent drawn from (6 ; –2) to point of intersection with the circle
$x^2 + y^2 - 6x + 2y + 8 = 0$

First we will write the equation of our circle in the form $(x - a)^2 + (y - b)^2 = r^2$ so that we can determine the co-ordinates of the centre of the circle and the radius.

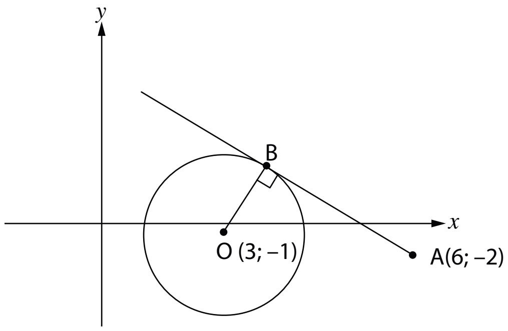
$\therefore\ x^2 - 6x + 9 + y^2 + 2y + 1 = -8 + 1 + 9$

$(x - 3)^2 + (y + 1)^2 = 2$

centre (3 ; –1)

radius $\sqrt{2}$

Now we need to draw a rough diagram.



We need to find the distance AB.

We know the OB = $\sqrt{2}$ units as OB is the radius of the circle.

We can find the distance formula

$$\begin{aligned} OA &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(3 - 6)^2 + (-1 - (-2))^2} \\ &= \sqrt{10} \end{aligned}$$

As we know that $\hat{B} = 90°$ (radius perpendicular to tangent) we can now use Pythagorus to solve for AB

$$\begin{aligned} OA^2 &= OB^2 + AB^2 \\ (\sqrt{10})^2 &= (\sqrt{2})^2 + AB^2 \\ AB^2 &= 8 \\ AB &= \sqrt{8} = 2\sqrt{2} \end{aligned}$$

Let's do another example.

Example

Example 2

Determine the length of the tangent drawn from A (4 , 5) to the point of contact with the circle.

$x^2 - 6x + y^2 + 10y + 14 = 0$

Once again we get the equation of our circle in the form $(x - a)^2 + (y - 6)^2 = r^2$

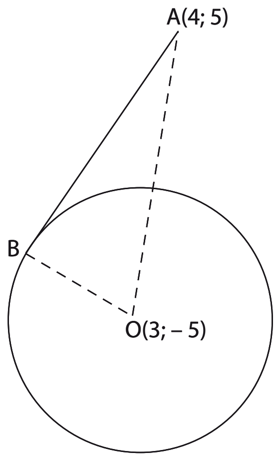
$x^2 - 6x + 9 + y^2 + 10y + 25 = -14 + 9 + 25$

$(x - 3)^2 + (y + 5)^2 = 20$

centre (3 ; –5)

radius = $\sqrt{20}$

Again we draw a rough diagram



We call the centre of the circle O and the point of contact of the tangent B.

We know the distance OB = $\sqrt{20}$ as OB is the radius of the circle.

The distance OA can be calculated using the distance formula.

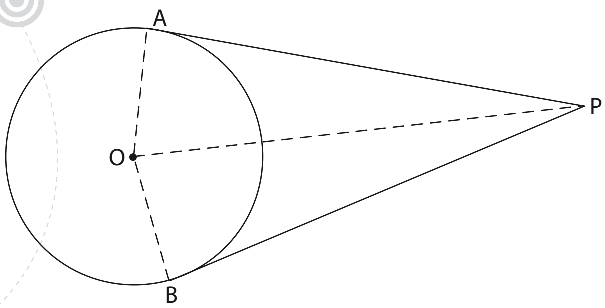
$$\begin{aligned} OA &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(3-4)^2 + (-5-5)^2} \\ &= \sqrt{101} \end{aligned}$$

As $\hat{B} = 90°$ we can use pythagorus to find AB

$$\begin{aligned} AB^2 + OB^2 &= AO^2 \\ AB + (\sqrt{20})^2 &= (\sqrt{101})^2 \\ AB^2 &= 81 \\ AB &= \sqrt{81} \end{aligned}$$

Let's have a look at some other important facts regarding circles and tangents.

1. Tangents drawn from a common point are equal in length.



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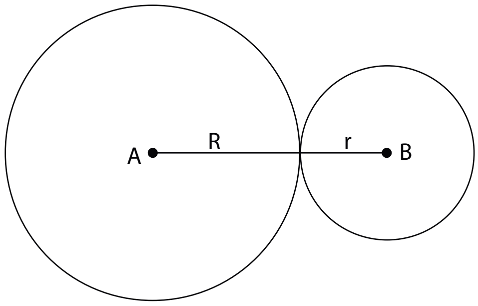
PA = PB

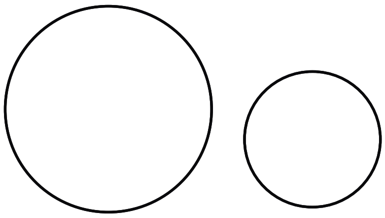
[Prove ΔPAO ≡ ΔPBO]

2. How to prove that 2 circles touch each other

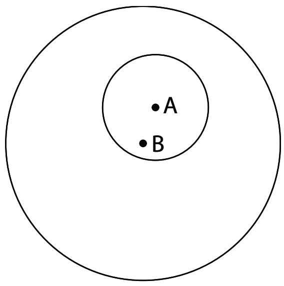


Let the centre of the one circle be A and the other B.

- Calculate the distance AB using the distance formula.
- Then add R (the radius of the one circle) to r the radius of the other. If AB = R+r the two circles touch each other and if AB > R + r the two circles never touch and if AB < R+ r the circles generally intersect at two points.



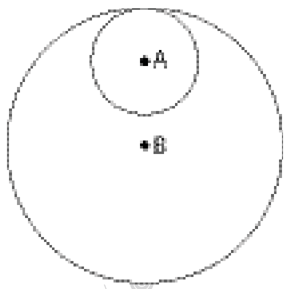
Let's look at the case of one circle inside another.



A is the centre of the smaller circle.
B is the centre of the bigger circle.
The distance AB is smaller than the sum of the radii.

In this example this does not mean the circles intersect as the smaller circle is is inside the bigger circle.
(A simultaneous equation between the two circles will yield no solution)

Have a look a the diagram alongside.



The distance between the centres AB ≠ r + R (the sum of the radii). In this case AB = R − r.
(Simultaneous equations will yield one solution)

The skills you have learnt in this section must now be put to the test.

Activity

Activity 3

1. The equation of a circle is given by: $x^2 + y^2 + 6x - 8y = 33$

1.1 Determine the co-ordinates of the centre E as well as the radius.

1.2 The line $y = x + 3$ cuts the circle in 2 points show that P(4 ; 7) is one of the points.

1.3 Determine the equation of the tangent to the circle at P.

2. A (3,7) B (7 ; –1) C $(-2 ; -\frac{1}{2})$ and D (7 ; a) are the points in a Cartesian plane. AB and CD intersect at 90°.

2.1 Show by calculation that $a = 4$.

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2.2 By calculation, show that if F is the midpoint of AB then C, F and D are collinear.

3. A circle with centre M (5 ; 4) and radius 5 units cuts the x axis at A and B with $x_A > x_B$

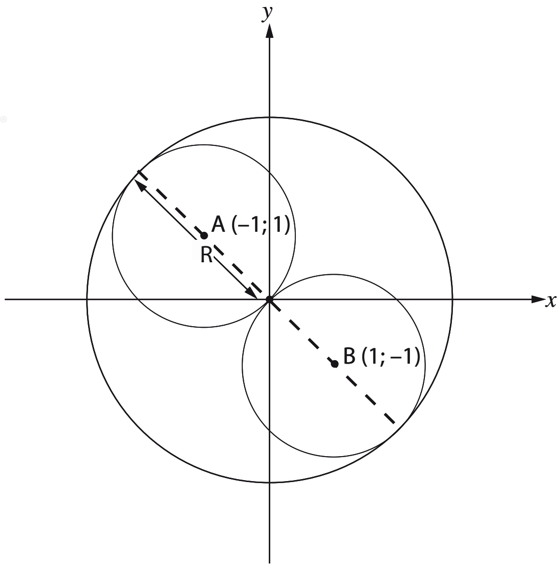
3.1 Write down the equation of the circle.

3.2 Find the co-ordinates of A and B.

3.3 Determine the equation of the tangent to the circle at A.

4. Given the circles: $x^2 + y^2 + 2x - 6y + 9 = 0$ and $x^2 + y^2 - 4x - 6y + 9 = 0$ show that the 2 circles touch externally.

5.



Consider the diagram above (Based on the yin yang sign)

The equation of the bigger circle is given by $x^2 + y^2 = 8$.

The smaller circles are centred at A and B respectively.

5.1 Give the equations of circles centred at A and B respectively.

5.2 Prove that circle centred A and the bigger circle touch internally.

5.3 Give the new equation of the bigger circle if it is translated 2 up and 3 left.

Solutions to Activities

Activity 1

1. $x^2 + y^2 = (\sqrt{3})$

   $x^2 + y^2 = 3$

2. $x^2 + y^2 = r^2$

   $(-5)^2 + (12)^2 = r^2$

   $r^2 = 169$

   $x^2 + y^2 = 169$

3. $x^2 + y^2 = r^2$

   $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = r^2$

   $r^2 = \frac{1}{2}$

   $x^2 + y^2 = \frac{1}{2}$

Activity 2

1. $x^2 - x + \left(\frac{1}{2}\right)^2 + y^2 - 2y + 1^2 = \left(\frac{1}{2}\right)^2 + 5 + 1^2$

   $(x - \frac{1}{2})^2 + (y - 1)^2 = \frac{25}{4}$

   centre $(\frac{1}{2}; 1)$ radius $\frac{5}{2}$

2. $x^2 + 2x + 1^2 + y^2 - 6y + 3^2 = -9 + 9 + 1$

$(x + 1)^2 + (y - 3)^2 = 1$

centre (–1 ; 3) radius 1

3. $x^2 - 4x + 2^2 + y^2 - 6y + 3^2 = -9 + 9 + 1$

$(x - 2)^2 + (y - 3)^2 = 4$

centre (2 ; 3) radius 2

Activity 3

1.1 $x^2 + 6x + 9 + y^2 - 8y + 16 = 33 + 9 + 16$

$(x + 3)^2 + (y - 4)^2 = 58$

centre E (–3 ; 4)

radius $\sqrt{58}$

1.2 $y = x + 3 \ldots (1)$

$x^2 + y^2 + 6x - 8y = 33 \ldots (2)$

sub (1) into (2)

$x^2 + (x + 3)^2 + 6x - 8(x + 3) = 33$

$x^2 + x^2 + 6x + 9 + 6x \;\; -8x - 24 - 33 = 0$

$2x^2 + 4x - 48 = 0$

$x^2 + 2x - 24 = 0$

$(x + 6)(x - 4) = 0$

$x = -6 \quad x = 4$

Sub $x = 4$ into $y = x + 3$

$y = 7$

P (4 ; 7)

1.3 $m_{radius} = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{7 - 4}{4 - (-3)} = \frac{3}{7}$

$m_{tangent} = \frac{-7}{3}$

$y - y_1 = m(x - x_1)$

$(y - 7) = -\frac{7}{3}(x - 4)$

$3y - 21 = -7x + 28$

$3y = -7x \; +49$

2.1 $m_{AB} = \frac{-1 - 7}{7 - 3} \;\; = \frac{-8}{4} \;\; = -2$

$m_{CD} = \frac{a - \left(-\frac{1}{2}\right)}{7 - (-2)} = \frac{a + \frac{1}{2}}{9}$

$m_{AB} \times m_{CD} = -1$

AB + CD

$\therefore \text{mcd} = \frac{1}{2}$

$$\frac{a + \frac{1}{2}}{a} = \frac{1}{2}$$

a = 4

2.2 $F\left(\frac{x_1 + x_2}{2} ; \frac{y_1 + y_2}{2}\right)$

$F\left(\frac{3+7}{2} ; \frac{7 + (-1)}{2}\right)$

F (5;3).

$M_{DF} = \frac{7-5}{4-3} = \frac{2}{1}$

$M_{DF} = \frac{4-(-\frac{1}{2})}{7-(-2)} = \frac{2}{1}$

$M_{DF} = M_{CD} \quad \therefore$

CFD colinear

3.1 $(x - 5)^2 + (y - 4)^2 = 25$

3.2 x int make y = 0

$(x - 5)^2 + (0-4)^2 = 25$

$x2 - 10x + 25 + 16 = 25$

$x^2 - 10x + 16 = 0$

$(x - 8)(x - 2)$

$x = 8 \quad x = 2$

If $x_A > x_B$

Then A (8 ; 0) and B (2 , 0)

3.3 $M_{radius} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-0}{5-2} = \frac{4}{3}$

$M_{tangent} = \frac{-3}{4}$

$y - y_1 = m(x - x_1)$

$y - 0 = -\frac{3}{4}(x - 2)$

$y = -\frac{3}{4}x + \frac{6}{4}$

$y = -\frac{3}{4}x = \frac{3}{2}$

4. $x^2 + y^2 + 2x - 6y + 9 = 0$

$(x + 1)^2 + (y - 3)^2 = 1$

Centre (–1 ; 3)

Radius $(r) = 1$

$x^2 + y^2 - 4x - 6y + 9 = 0$

$(x-2)^2 + (y-3)^2 = 4$

Centre (2 ; 3)

Radius (R) = 2

$\therefore r + R = 2$

Distance between (–1 ; 3) and (2 ; 3) is 3 = R + r

$\therefore$ Circles touch externally

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5.1 Circle Centre A: $(x + 1)^2 + (y - 1)^2 = 2$

Circle Centre B: $(x - 1)^2 + (y + 1)^2 = 2$

5.2 distance AO = $\sqrt{(-1 - 0) + (1 - 0)^2}$

$= \sqrt{2}$

distance BO = $\sqrt{2}$

Distance between centres

AO = $\sqrt{2}$

radius of bigger circle $2\sqrt{2}$

radius of smaller circle $\sqrt{2}$

OA = $2\sqrt{2} - \sqrt{2} = \sqrt{2}$

$\therefore$ touch internally

5.3 $(x + 3)^2 + (y - 2)^2 = 8$.