A teacher ...
takes a hand
opens a mind &
touches a heart

THE ANSWER SERIES
Your Key to Exam Success

MATHS TEACHER SUPPORT:
Problem-solving in FET Analytical Geometry

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Problem-solving in FET Analytical Geometry

14 March 2022

Hosted by Gretel Lampe

Presented by Anne Eadie

Q & A by Jenny Campbell & Susan Carletti

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Analytical Geometry Question 1

1.1 Determine the maximum radius of the circle with equation

$$x^2 + y^2 + 6x\cos\theta - 4y\sin\theta + 3 = 0$$

Analytical Geometry Question 1 Solution

1.1 Determine the maximum radius of the circle with equation

$$x^2 + y^2 + 6x\cos\theta - 4y\sin\theta + 3 = 0$$

$$x^2 + y^2 + 6x\cos\theta - 4y\sin\theta + 3 = 0$$

$$\therefore \ (x + 3\cos\theta)^2 + (y - 2\sin\theta)^2 = -3 + 9\cos^2\theta + 4\sin^2\theta$$

$$\therefore \ r^2 = -3 + 9\cos^2\theta + 4\sin^2\theta$$

$$\therefore \ r^2 = -3 + 9\left(1 - \sin^2\theta\right) + 4\sin^2\theta$$

$$\therefore \ r^2 = 6 - 5\sin^2\theta$$

But $0 \leq \sin^2\theta \leq 1$

$\therefore$ max $r^2 = 6 - 0 = 6$

$\therefore$ max $r = \sqrt{6}$

or

$$\therefore \ r^2 = -3 + 9\cos^2\theta + 4\left(1 - \cos^2\theta\right)$$

$$r^2 = 1 + 5\cos^2\theta$$

But $0 \leq \cos^2\theta \leq 1$

$\therefore$ max $r^2 = 1 + 5 = 6$

$\therefore$ max $r = \sqrt{6}$

Analytical Geometry Question 1

$$x^2 + y^2 + 6x\cos\theta - 4y\sin\theta + 3 = 0$$

1.2 For this maximum radius, determine the coordinates of the centre(s) of the circle(s).

Analytical Geometry Question 1 Solution

$$x^2 + y^2 + 6x\cos\theta - 4y\sin\theta + 3 = 0$$

1.2 For this maximum radius, determine the coordinates of the centre(s) of the circle(s).

$$(x + 3\cos\theta)^2 + (y - 2\sin\theta)^2 = r^2$$

For maximum radius:

$r^2 = 6 - 5\sin^2\theta$

$\therefore \sin^2\theta = 0$

$\therefore \sin\theta = 0$

$\therefore \theta = 0^\circ + n180^\circ$

or

$r^2 = 1 + 5\cos^2\theta$

$\therefore \cos^2\theta = 1$

$\therefore \cos\theta = \pm 1$

centre $= (-3\cos\theta;\ 2\sin\theta)$

If $\theta = 0^\circ$ centre $= (-3; 0)$

If $\theta = 180^\circ$ centre $= (3; 0)$

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Problem-solving in FET Analytical Geometry

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CONTENT FRAMEWORK

- **Formulae**
- **The ∠ of inclination**
- **Straight line graphs**

} (Gr 10 & 11)

- **Circles** (Gr 12)

Pre-Grade 12

- Formulae & ∠ of Inclination
- Graph Concepts:
  Equations of Straight Line Graphs
- Euclidean Geometry:
  Lines, Δ^s, Quadrilaterals

Grade 12

CIRCLES

- Equations of Circles
  - General and Standard Form
- A Tangent to a Circle

Points of Intersection

- Circle and a line
- 2 Circles

ANALYTICAL GEOMETRY TOOLKIT

Refer to the Answer Series Gr 12 Maths 2 in 1 pages xiii & xiv

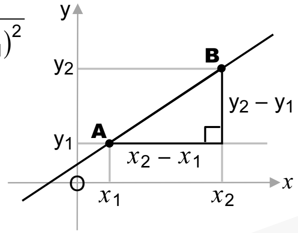
FORMULAE

Consider two points A(x_1; y_1) and B(x_2; y_2):

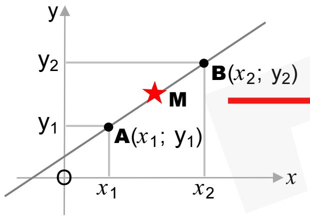
DISTANCE

$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ . . . *Thm of Pythagoras*

$\therefore$ **AB** $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



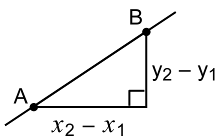
MIDPOINT



$M\left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$

The co-ordinates of the midpoint, M, are the ***averages*** *of the co-ordinates of the endpoints, A and B.*

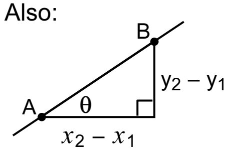
GRADIENT



$m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$

. . . *the* ***gradient*** *of the line*

Also:

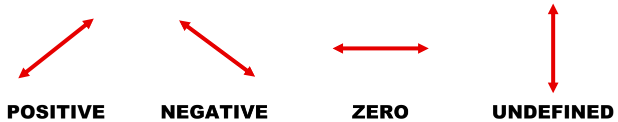


$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y_2 - y_1}{x_2 - x_1}$

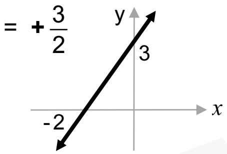
. . . *where θ is the* ***angle of inclination*** *of the line*

The Gradient of a line ↔ The Inclination of a line

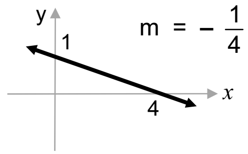
Values



$m = +\frac{3}{2}$

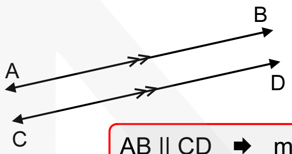


$m = -\frac{1}{4}$



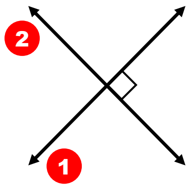
Parallel lines

Parallel lines have **equal** gradients.



AB || CD ➡ $m_{AB} = m_{CD}$

Perpendicular lines

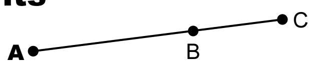


If the gradient of line ① is $\frac{2}{3}$, then the gradient of line ② will be $-\frac{3}{2}$

Note: $m_1 \times m_2 = \left(+\frac{2}{3}\right)\left(-\frac{3}{2}\right) = -1$

i.e. The **product** of the gradients of ⊥ lines is **–1**.

Collinear points



Three points A, B & C are collinear if the gradients of **AB** & **AC** are equal. (Note: Point **A** is common.)

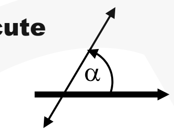
$m_{AB} = m_{AC}$ ⬌ A, B & C are collinear

The Inclination of a line

Angles α and β below are **angles of inclination**.

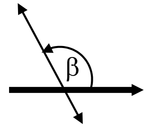
The inclination of a line is the **angle** which the line makes with the positive direction of the x-axis.

α **acute**



gradient of line is positive

β **obtuse**



gradient of line is negative

Gradient, **m = tan** α or **tan** β where α and β are the ∠**s** **of inclination**.

GRAPH CONCEPTS

1 : Axis intercepts

Every point on the **y-axis** has **x = 0**.
Every point on the **x-axis** has **y = 0**.

2 : The equation

The **equation** of a graph is true for **all** points on the graph.

∴ The **equation** of the **y-axis** is **x = 0**;
& the **equation** of the **x-axis** is **y = 0**.

3 : Types of graph

Different **types/patterns** are indicated by various equations.

e.g. $\mathbf{y = mx + c}$ indicates a straight line

$\mathbf{x^2 + y^2 = r^2}$ indicates a circle

$y = ax^2 + bx + c$ indicating a parabola

GRAPH CONCEPTS cont . . .

FACT 1 : Points on Graphs

If a point lies on a graph, the equation is true for its coordinates, i.e. the coordinates of the point satisfy the equation . . . so, substitute!

and, conversely,

If a point (i.e. its coordinates) satisfies the equation of a graph (i.e. "makes it true"), then it lies on the graph.

FACT 2 : Point(s) of Intersection

The coordinates of the point(s) of intersection of two graphs "obey the conditions" of both graphs, i.e. they satisfy both equations simultaneously.

They are found:

- **"algebraically"** by solving the 2 equations, or
- **"graphically"** by reading from the graph.

THESE 2 FACTS ARE CRUCIAL!

STRAIGHT LINE GRAPHS & their equations

Standard forms

- **$y = mx + c$:**

  where **m** = the gradient & **c** = the y-intercept

  When m = 0: **$y = c$** ... *a line* || **x-axis** ... **HORIZONTAL LINES**

  When c = 0: **$y = mx$** ... *a line through the* **origin**

  Also: **$x = k$** ... *a line* || **y-axis** ... **VERTICAL LINES**

- **$y - y_1 = m(x - x_1)$:**

  where **m** = the gradient & **$(x_1; y_1)$** is a fixed point.

General form

The **general form** of the equation of a straight line is **$ax + by + c = 0$**, e.g. $2x + 3y + 6 = 0$

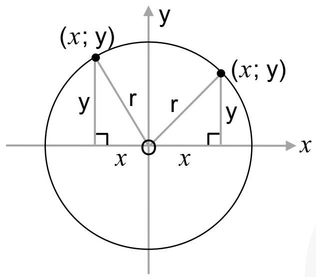
CIRCLES & their equations

Circles with the origin as centre

True of **any** point **$(x; y)$** on a circle with centre **(0; 0)** and radius **r** is that:

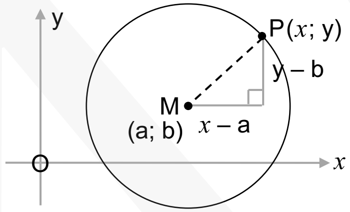
$$x^2 + y^2 = r^2$$

Thm. of Pythag.!



Circles with any given centre

True of any point **$(x; y)$** on a circle with centre **$(a; b)$** and radius **r** is that:



$$(x - a)^2 + (y - b)^2 = r^2$$

Distance formula! (Thm. of Pythag.)

Converting the equation of a circle

General form: $Ax^2 + Bx + Cy^2 + Dy + E = 0$

to **Standard form:** $(x - a)^2 + (y - b)^2 = r^2$

(using completion of squares)

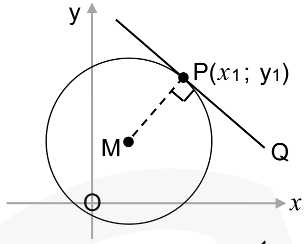
e.g.
$$x^2 - 6x + y^2 + 8y - 25 = 0$$
$$\therefore x^2 - 6x + y^2 + 8y = 25$$
$$\therefore x^2 - 6x + \mathbf{3^2} + y^2 + 8y + \mathbf{4^2} = 25 + \mathbf{9} + \mathbf{16}$$
$$\therefore (x - 3)^2 + (y + 4)^2 = 50$$

This is the equation of a circle with:

centre $(3; -4)$ & **radius**, $r = \sqrt{50}\ (= 5\sqrt{2})$ units

A Tangent to a circle . . .

is **perpendicular** to the **radius** of the circle at the **point** of **contact**.



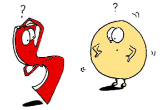
$m_{MP} = 2 \Rightarrow m_{PQ} = -\frac{1}{2}$

($\because$ radius MP $\perp$ tangent PQ)

To find the **equation** of a tangent, use **"m and 1 point"** in the straight line equation:

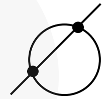
$$y - y_1 = m(x - x_1)$$

Point(s) of intersection of a Line and a Circle

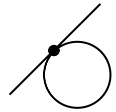


A line and a circle **either**

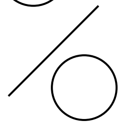
1. **"cut"** (twice!) [secant] (2 points in common)



or 2. **"touch"** (once!) [tangent] (1 point in common)



or 3. **don't cut or touch** (no points in common)



If we substitute $y = mx + c$ into the equation of the ⊙, there will **either** be: 1 **2 solutions** **or** 2 **1 solution** **or** 3 **no solutions** for x, resulting in one of the above scenarios.

FINAL ADVICE

Use common sense

& ALWAYS DRAW A PICTURE!!!

TAS FET ANALYTICAL GEOMETRY COURSE

Teaching Documents

by The TAS Maths Team

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TEACHING DOCUMENTS

- Exam Mark Distribution (Maths Paper 2)
- The 2022 ATP (Proposed)
- Research: Results & Diagnostic Reports (2020 & 2021)
  including Exam questions and memos
- Exemplar Analytical Geometry Questions and Detailed Solutions
- The CAPS Curriculum Overview

FET EXAM: Mark distribution

PAPER 2 Proofs: maximum 12 marks			
Description	**GR 10**	**GR 11**	**GR 12**
Statistics	15	20	20
Analytical Geometry	**15**	**30**	**40**
Trigonometry	40	50	50
Euclidean Geometry & Measurement	30	50	40
TOTAL	**100**	**150**	**150**

NOTE:

- Questions will not necessarily be compartmentalised in sections, as this table indicates. Various topics can be integrated in the same question.
- A formula sheet will be provided for the final examinations in Grades 10, 11 and 12.

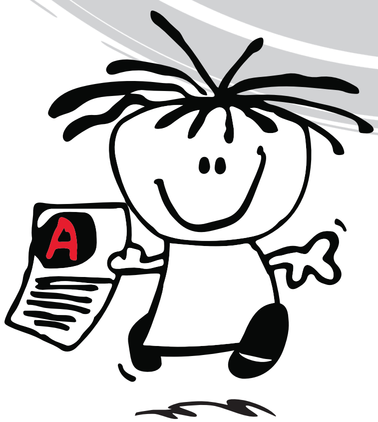
January 2022

A PROPOSED 2022 ATP FOR FET MATHS

	Grade 10		Grade 11		Grade 12	
		No. of weeks		**No. of weeks**		**No. of weeks**
Term 1	Algebraic Expressions, Numbers & Surds	4	Exponents & Surds	1	Patterns, Sequences and Series	3
	Exponents, Equations & Inequalities	2	Equations	1	Euclidean Geometry	3
	Equations & Inequalities	1	Equations & Inequalities	2	Trigonometry	4
	Euclidean Geometry (#1)	3	Euclidean Geometry	4	(Algebra)	
			Trig functions & Revision of Gr 10 Trig	1		
			Trig identities & Reduction formulae	1		
Term 2	Trigonometry (#1)	3	Trig eqn. & Gen. sol's	1	**Analytical Geometry**	**2**
	Number Patterns	1	Quadrilaterals	1	Functions & Inverse Functions & Exp & Log Functions	2
	Functions (including Trig Functions (#2))	6	**Analytical Geometry**	**2**	Calculus, including Polynomials	5
	Measurement	2	Number Patterns	2	Finance	3
			Functions	5		
			Trig – sin/cos/area rules	1		
Term 3	Statistics	2	Trig – sin/cos/area rules	1	Finance	1
	Probability	2	Measurement	2	Statistics (regression & correlation)	3
	Finance (Growth)	2	Statistics	3	Counting and Probability	3
	Analytical Geometry	**2**	Probability	4	**INTERNAL EXAMS**	**4**
	Euclidean Geometry (#2)	3	Finance (Growth & Decay)	1		
Term 4	Revision	4	Finance (Growth & Decay)	3	Revision (Paper 1)	1
	FINAL EXAMS	3	Revision	1	Revision (Paper 2)	1
	Reporting	1 ½	**FINAL EXAMS**	**4**	Revision (Exam Techniques?)	1
			Reporting	1½	**EXTERNAL EXAMS**	6½

RESEARCH:

Results & Diagnostic Reports



Some interesting statistics ...

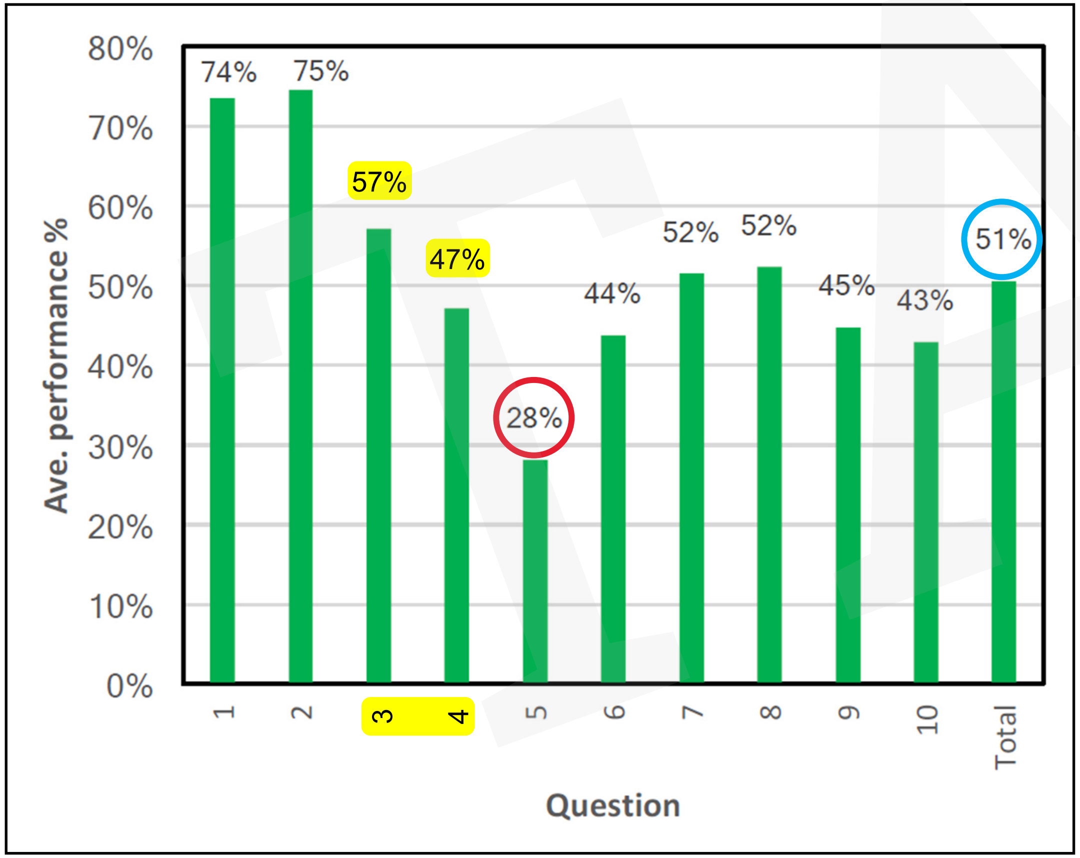
PAPER 2:	ave. over 8 years **2014 – 2021**	**2019**	**2020**	**2021**
Statistics	61,9%	61,4% ↓	74,5% ↑	75% ↑
Analytical Geometry	53,9%	57,7% ↑	52,3% ↓	51,6% ↓
Trigonometry	41,3%	34,6% ↓	42,9% ↑	39,4% ↓
Euclidean Geometry	44,6%	45,7% ↑	46,5% ↑	33,5% ↓
Paper 2	50%	48% ↓	51% ↑	46% ↓
Paper 1	52%	52%	51% ↓	53% ↑

Note: This Rasch analysis is based on a random sample of candidates and may not reflect the national averages accurately.

However, it is useful in assessing the RELATIVE degrees of each question AS EXPERIENCED BY CANDIDATES.

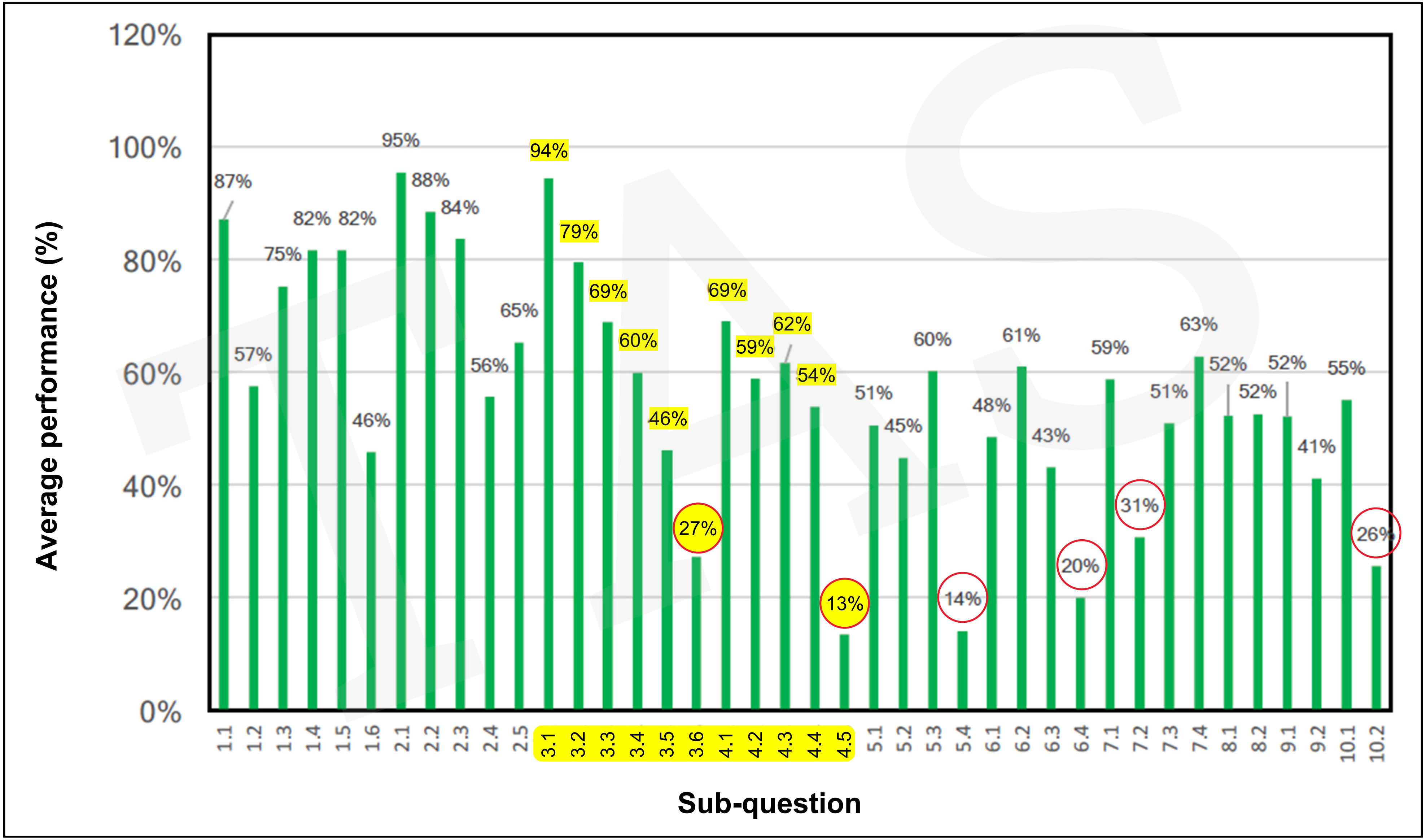
2020: Paper 2

Average % performance per question



Q1	Data Handling
Q2	Data Handling
Q3	Analytical Geometry
Q4	Analytical Geometry
Q5	Trigonometry
Q6	Trigonometry
Q7	Trigonometry
Q8	Euclidean Geometry
Q9	Euclidean Geometry
Q10	Euclidean Geometry

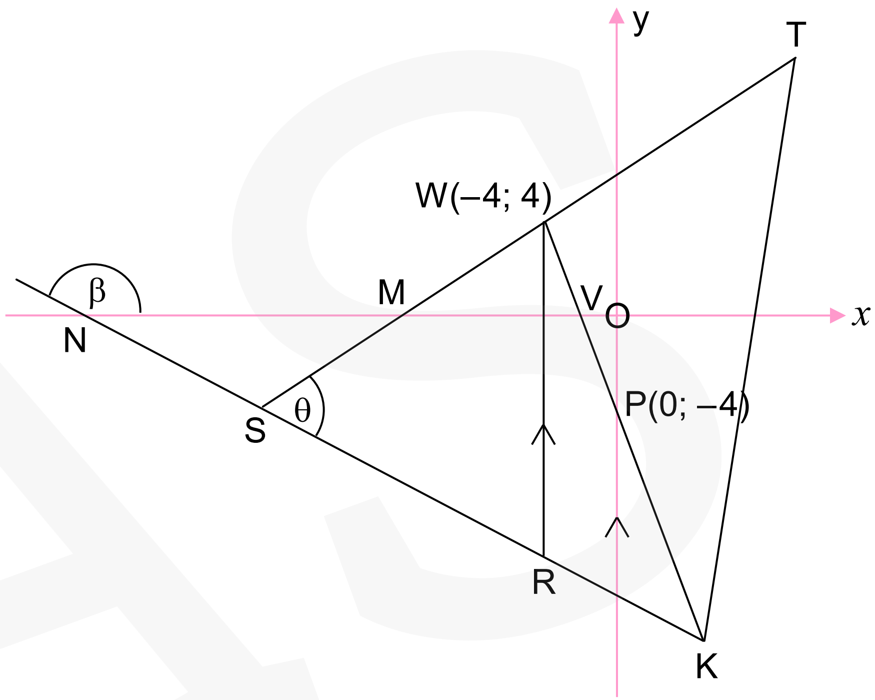
and, per sub-question



GR 12 NAT NOV 2020 – ANALYTICAL GEOMETRY

QUESTION 3   57%

- ΔTSK is drawn.
- The equation of ST is $y = \frac{1}{2}x + 6$ and ST cuts the x-axis at M.
- W(–4; 4) lies on ST and R lies on SK, such that WR is parallel to the y-axis.
- WK cuts the x-axis at V and the y-axis at P(0; –4).
- KS produced cuts the x-axis at N.
- $T\hat{S}K = \theta$.



94% 3.1 Calculate the gradient of WP. (2)

79% 3.2 Show that WP $\perp$ ST. (2)

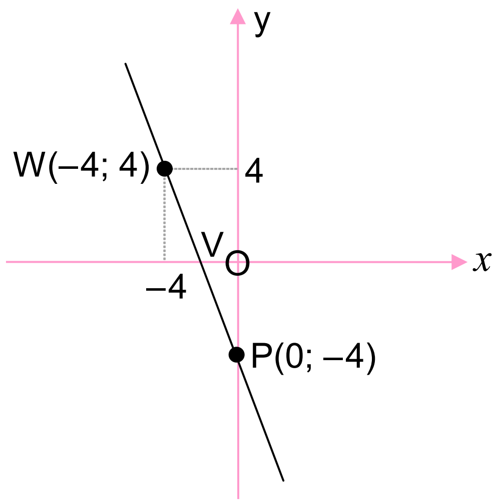
69% 3.3 If the equation of SK is given as $5y + 2x + 60 = 0$, calculate the coordinates of S. (4)

60% 3.4 Calculate the length of WR. (4)

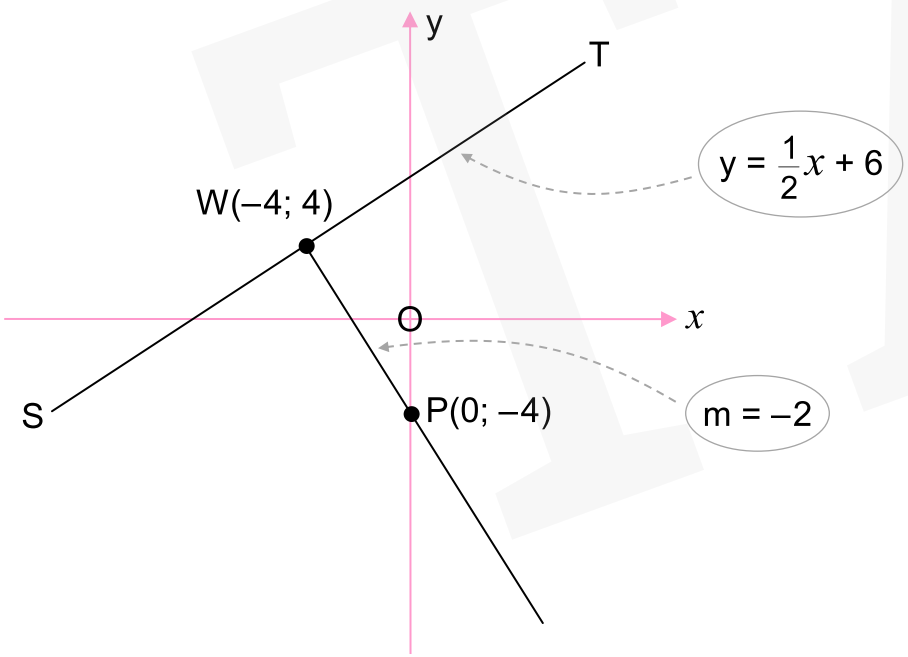
46% 3.5 Calculate the size of θ. (5)

27% 3.6 Let L be a point in the third quadrant such that SWRL, in that order, forms a parallelogram. Calculate the area of SWRL. (4) **[21]**

3.1 **m_{WP} ?**



3.2 **Show that WP ⊥ ST**

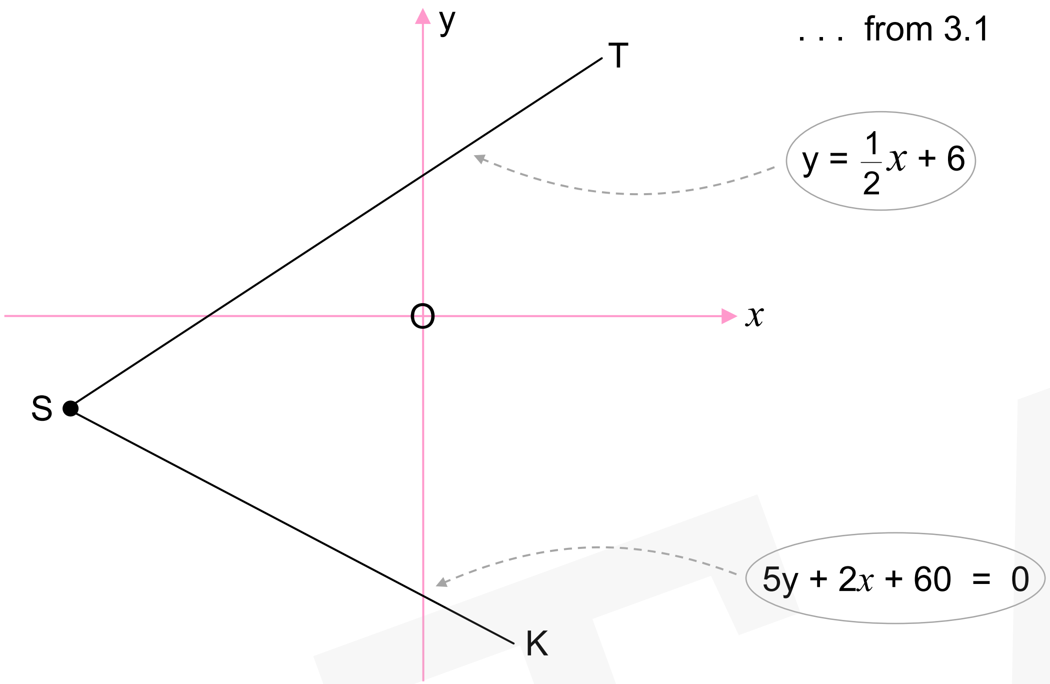


Common Errors and Misconceptions

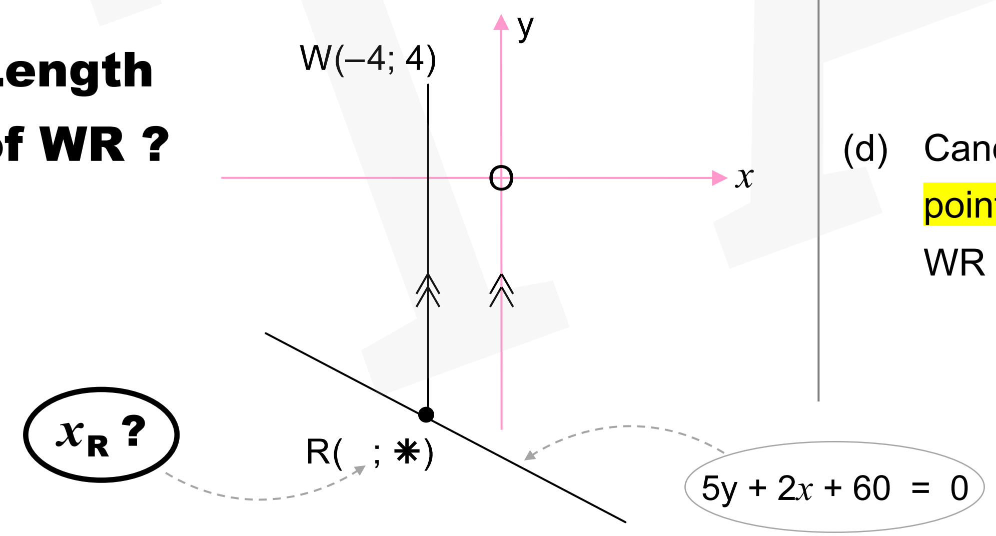
(a) In Q3.1 candidates made an incorrect substitution into the gradient formula, e.g. $m_{wp} = \frac{4-4}{-4-0}$.

(b) The equation of the straight line ST was given. Notwithstanding this, some candidates could not identify the gradient of ST in answering Q3.2. **Instead of proving** that ST was perpendicular to WP, these **candidates assumed** that WP was perpendicular to ST and then went on to calculate the gradient of ST.

3.3 **Coordinates of S ?**



3.4 **Length of WR ?**

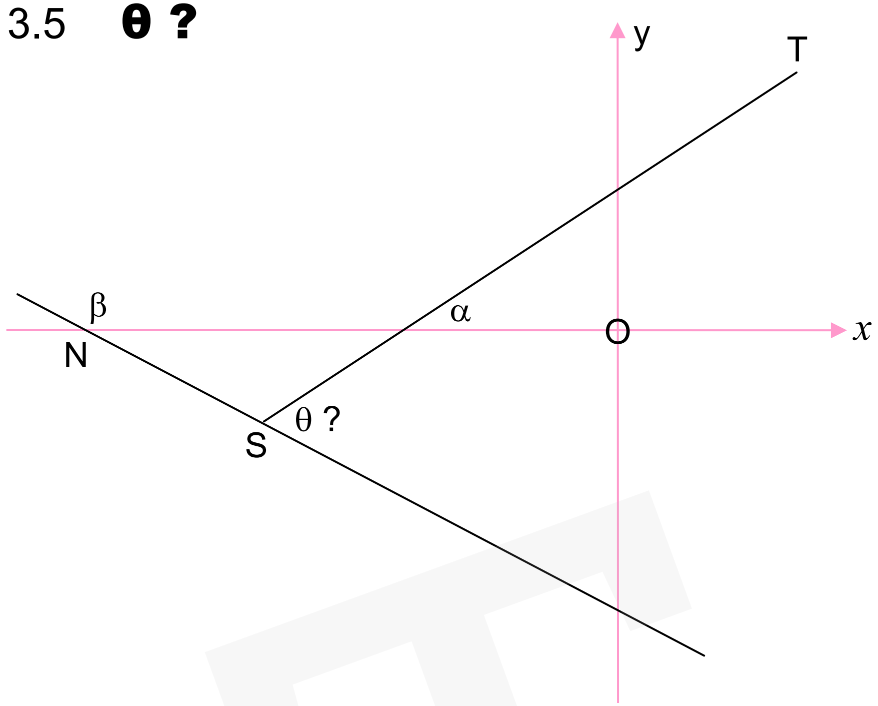


Common Errors and Misconceptions

(c) In answering Q3.3 some candidates had difficulty in rewriting the equation $5y + 2x + 60 = 0$ in y-form. Instead of solving a set of simultaneous equations to calculate the values of x and y, some candidates created the equation $5y + 2x + 60 = \frac{1}{2}x + 6$ and then went on to calculate the x- and y-intercepts of this function. Some candidates **incorrectly assumed** that SP was **parallel to the x-axis** and therefore the y-coordinate of S was –4. They then substituted this y-value into the equation $5y + 2x + 60 = 0$ to obtain the x-value of –20. Hence they arrived at the coordinates of S to be (–20 ; –4).

(d) Candidates could not identify the x-coordinate of the point R. Hence they could not calculate the length of WR in Q3.4.

3.5 **θ ?**

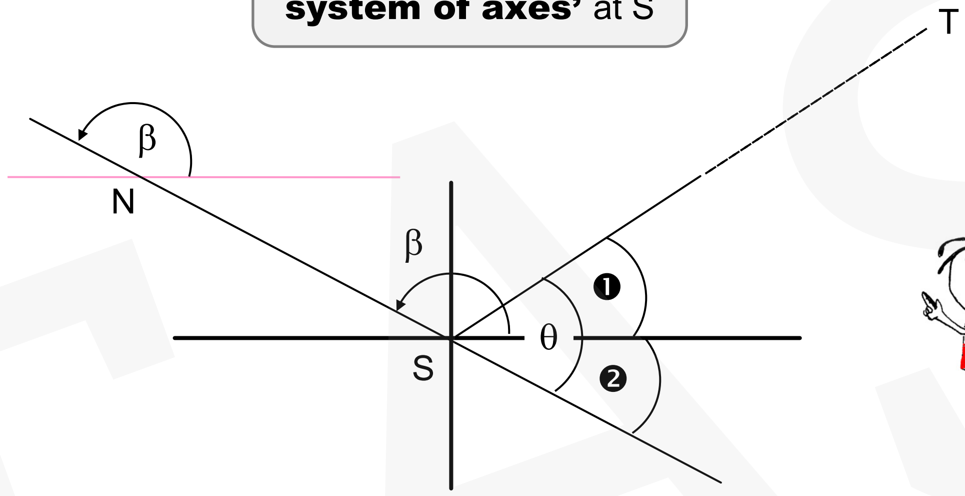


Common Errors and Misconceptions

(e) In Q3.5 many candidates were unable to use the formula tan θ = m correctly. They either used incorrect angles or incorrect gradients in the formula. Some candidates referred to all angles in their calculations as θ. This lead to major confusion.

OR: A nice technique!

We add in a **'mini system of axes'** at S

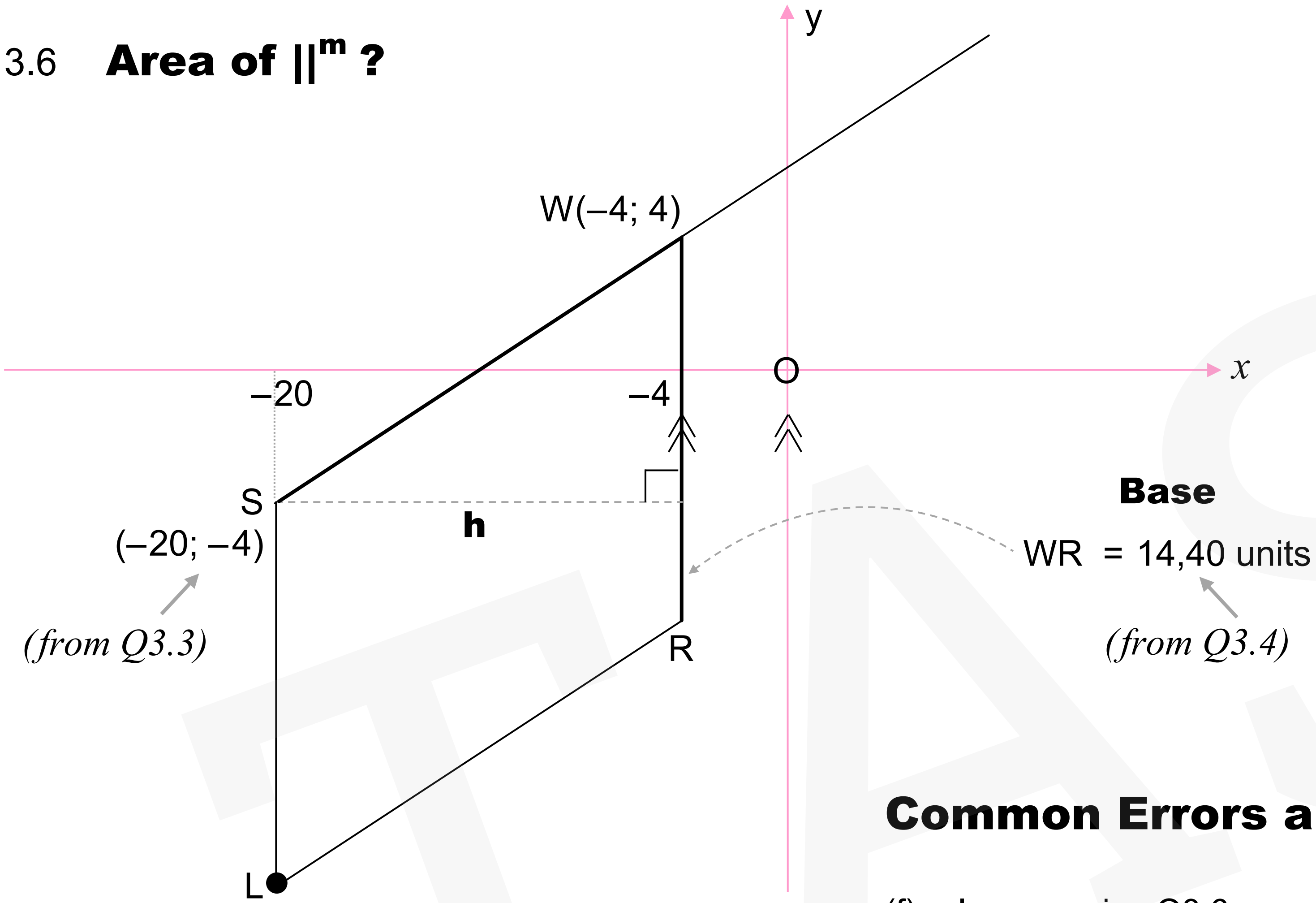


- tan β = $-\frac{2}{5}$ ➡ β = 180º – 21,801...º ➡ S_2 = 21,801...º

- tan S_1 = $\frac{1}{2}$ ➡ S_1 = 26,565...º

∴ θ = 26,565...º + 21,801...º ≈ **48,37º** ◄

3.6 **Area of $||^m$?**



Euclidean Geometry ($||^m$)

Area ?

Common Errors and Misconceptions

(f) In answering Q3.6, many candidates adopted the strategy of calculating the area of a triangle.

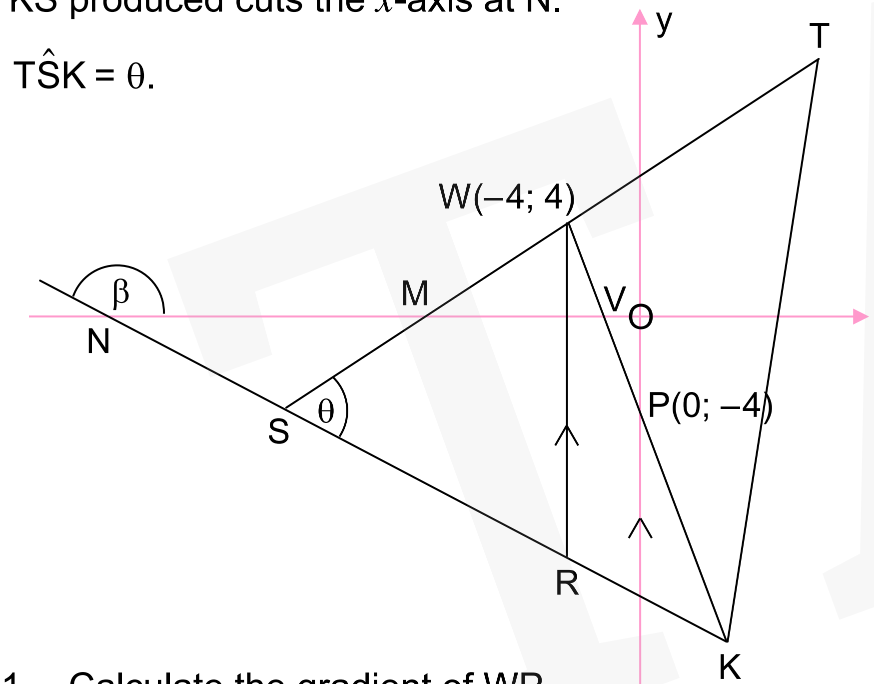
However, they used the formula:

area of triangle = $\frac{1}{2} base \times height$ even though the perpendicular height of the triangle was unknown.

They substituted the length of a side of the triangle as the perpendicular height.

QUESTION 3

- ΔTSK is drawn.
- The equation of ST is $y = \frac{1}{2}x + 6$ and ST cuts the x-axis at M.
- W(–4; 4) lies on ST and R lies on SK, such that WR is parallel to the y-axis.
- WK cuts the x-axis at V and the y-axis at P(0; –4).
- KS produced cuts the x-axis at N.
- $T\hat{S}K = \theta$.



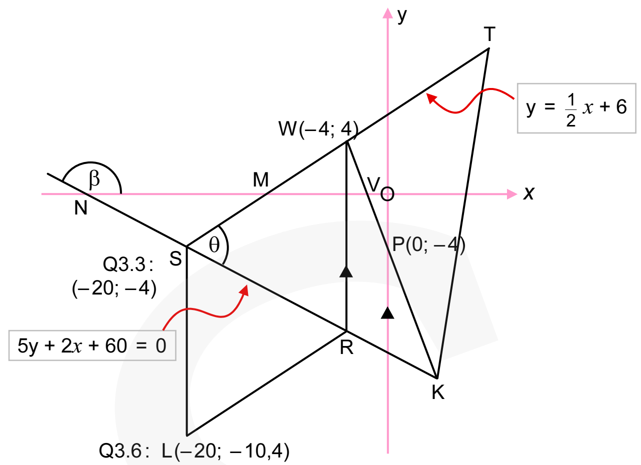
3.1 Calculate the gradient of WP. (2)

3.2 Show that WP ⊥ ST. (2)

3.3 If the equation of SK is given as $5y + 2x + 60 = 0$, calculate the coordinates of S. (4)

MEMO

3.



3.1 $m_{WP} = \frac{-4-4}{0-(-4)} = \frac{-8}{4} = \mathbf{-2}$ ✓

3.2 $m_{ST} = \mathbf{\frac{1}{2}}$ ✓ . . . *from equation*

$\therefore m_{WP} \times m_{ST} = (-2)\left(\frac{1}{2}\right) = -1$

∴ **WP ⊥ ST** ✓ . . . *product of gradients = –1*

3.3 At S: $y = \frac{1}{2}x + 6$ . . . ❶

and $5y + 2x + 60 = 0$ . . . ❷

❶ in ❷: $\therefore 5\left(\frac{1}{2}x + 6\right) + 2x + 60 = 0$

$\therefore \frac{5}{2}x + 30 + 2x + 60 = 0$

$\therefore \frac{9x}{2} = -90$

$\times\left(\frac{2}{9}\right) \quad \therefore x = -20$

❶: $y = \frac{1}{2}(-20) + 6 = -4$

∴ **Pt S (–20; –4)** ✓

(4)

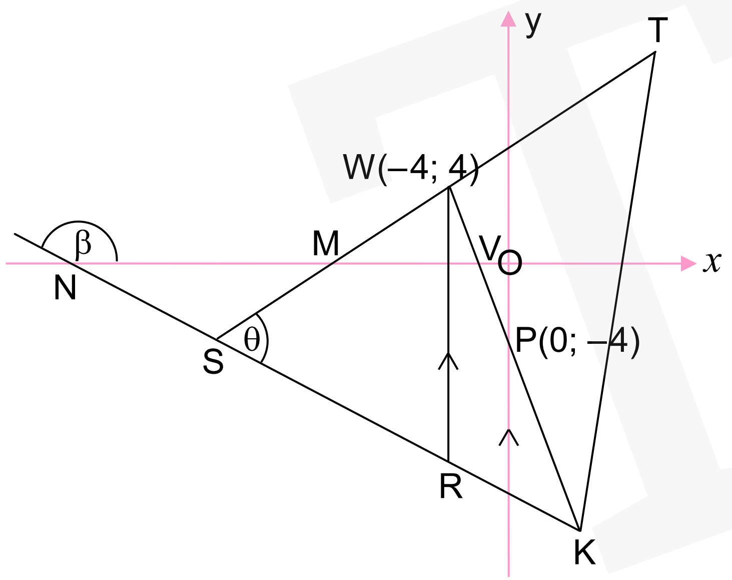
QUESTION 3 (cont.)

3.4 Calculate the length of WR. (4)

3.5 Calculate the size of θ. (5)

3.6 Let L be a point in the third quadrant such that SWRL, in that order, forms a parallelogram.

Calculate the area of SWRL. (4)



MEMO

3.4 WR $= y_W - y_R$ . . . *vert. length = difference of y-coords.*

At R: $x_R = x_W = -4$

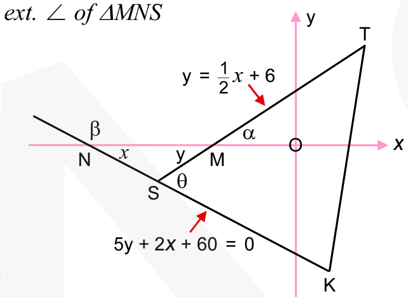
& $5y + 2(-4) + 60 = 0$

$\therefore 5y = -52$

$\therefore y = -\frac{52}{5}$

$\therefore$ **WR** $= 4 - \left(-\frac{52}{5}\right) = 14\frac{2}{5} =$ **14,40 units** ✓

3.5 $\theta = \hat{x} + \hat{y}$ . . . *ext. ∠ of ΔMNS*



Equation of SK:

$5y + 2x + 60 = 0$

$\therefore 5y = -2x - 60$

$\therefore y = -\frac{2}{5}x - 12$

- $\mathbf{m_{SK}} = -\frac{2}{5}$

  $\therefore \beta = 180° - \tan^{-1}\left(\frac{2}{5}\right)$

  $= 180° - 21{,}801...°$

  $\therefore x = 21{,}801...°$ . . . *∠s on straight line*

- $\mathbf{m_{ST}} = \frac{1}{2}$ . . . *gradient of ST*

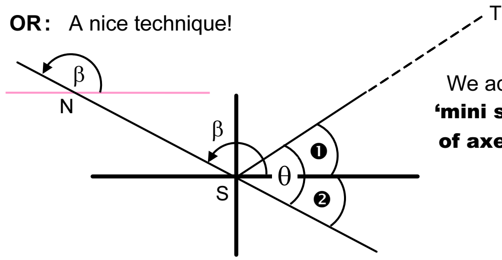
  $\therefore \alpha = \tan^{-1}\left(\frac{1}{2}\right) = 26{,}565...°$

  $\therefore y = 26{,}565...°$

$\therefore \theta = 21{,}801...° + 26{,}565...°$

$\approx$ **48,37°** ✓

OR: A nice technique!



We add in a **'mini system of axes'** at S

- $\tan\beta = -\frac{2}{5}$ ➡ $\beta = 180° - 21{,}801...°$ ➡ $S_2 = 21{,}801...°$
- $\tan S_1 = \frac{1}{2}$ ➡ $S_1 = 26{,}565...°$

  $\therefore \theta = 26{,}565...° + 21{,}801...° \approx$ **48,37°** ✓

3.6 **Area of ||^m = base × height**

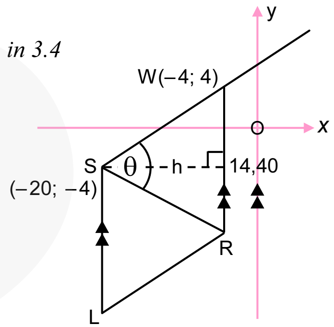
Base, WR = 14,40 units . . . *in 3.4*

& **Height**, h $= x_W - x_S$

$= -4 - (-20)$

$= 16$ units

The height (h) is a horizontal length here.



$\therefore$ Area = 14,40 × 16

= **230,40 units²** ✓

OR: Area of ΔWSR $= \frac{1}{2}$ SW.SR $\sin\theta$

& Area of ||^m SWRL = 2 × area of ΔWSR

Q3: General Suggestions for Improvement

(a) If learners are not sure, they should consult the information sheet for the correct formula.

(b) Substitution into the formula remains a problem. Learners should first write down the coordinates and then substitute them into the formula.

(c) Teachers should request learners to label the coordinates as $(x_1 ; y_1)$ and $(x_2 ; y_2)$ on the diagram. This should prevent learners from making mistakes when substituting the coordinates into a formula. The order of substitution must be consistent, especially when using the gradient formula.

(d) Teachers should encourage learners to write down the values that they have already calculated (lengths, angles and gradients) on the diagram. This will assist learners when answering follow-up questions. Learners should label different angles using different symbols, e.g. α, β, θ, etc.

(e) Candidates must be made aware that when the questions say **'show that'**, the answer is already there. Their **task is to prove** that the **statement is true**.

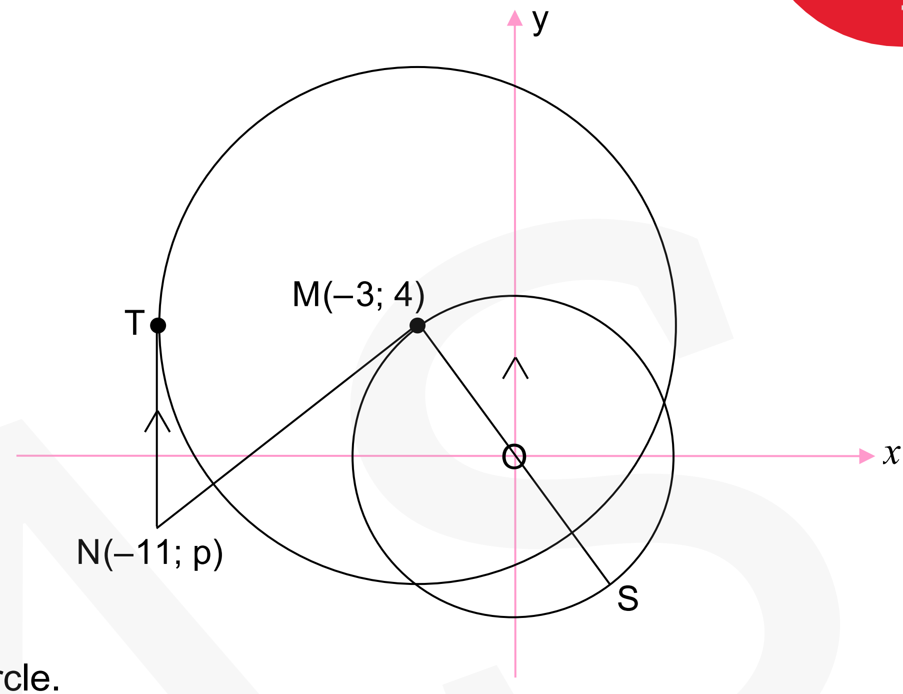
(f) To answer questions in analytical geometry well, learners should master the properties of quadrilaterals and triangles. Constant revision of Analytical Geometry concepts taught in Grades 10 and 11 is essential, as much of the Grade 12 work revolves around these concepts.

(g) Learners should refrain from making assumptions about features in a question. These need to be proved first before the results can be used in an answer.

(h) The **different topics in Mathematics should be integrated**. Learners must be able to establish the **connection between Euclidean Geometry and Analytical Geometry**.

QUESTION 4 47%

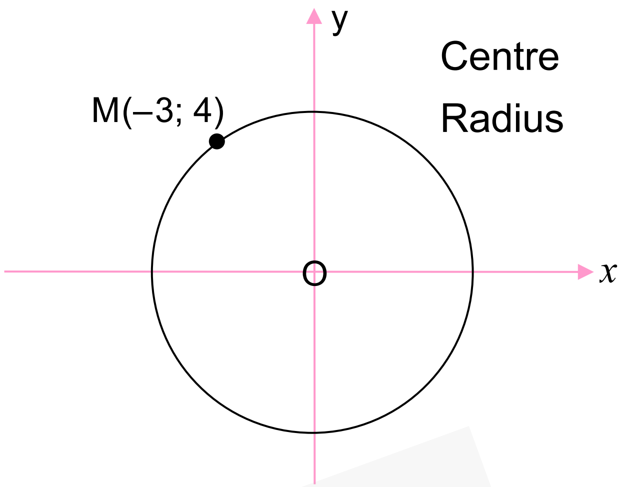
- M(–3; 4) is the centre of the large circle and a point on the small circle having centre O(0; 0).
- From N(–11; p), a tangent is drawn to touch the large circle at T with NT parallel to the y-axis.
- NM is tangent to the smaller circle at M with MOS a diameter.



69%	4.1	Determine the equation of the small circle.	(2)
59%	4.2	Determine the equation of the circle centred at M in the form $(x - a)^2 + (y - b)^2 = r^2$	(3)
62%	4.3	Determine the equation of NM in the form $y = mx + c$.	(4)
54%	4.4	Calculate the length of SN.	(5)
13%	4.5	If another circle with centre B(–2; 5) and radius k touches the circle centred at M, determine the value(s) of k, correct to ONE decimal place.	(5)

[19]

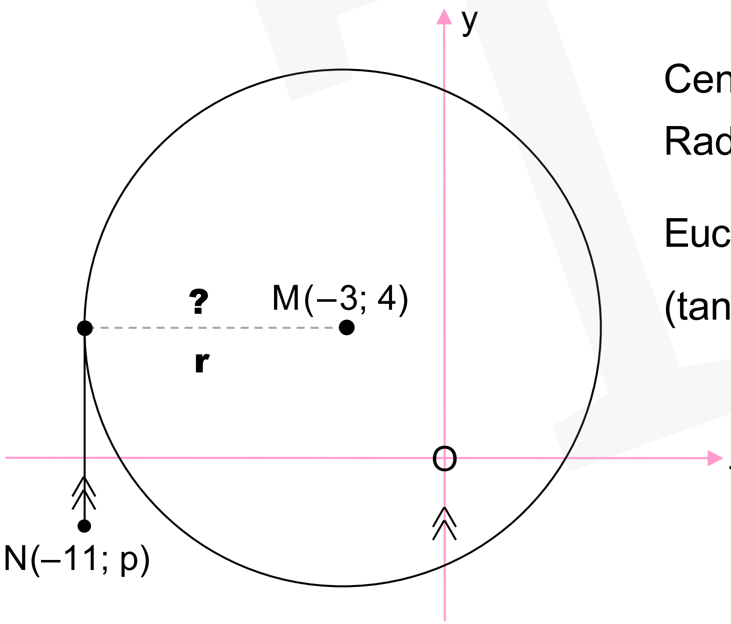
4.1 **Equation of ⊙O?**



Centre

Radius

4.2 **Equation of ⊙M?**



Centre

Radius

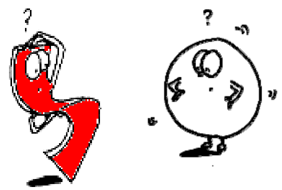
Euclidean Geometry

(tangent ⊥ radius)

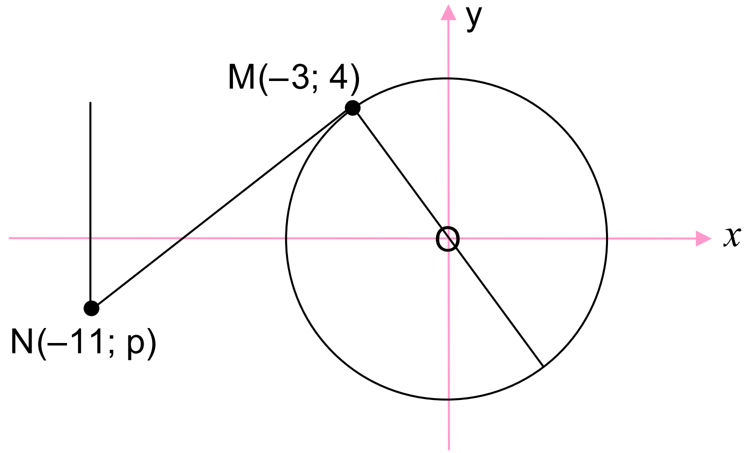
Common Errors and Misconceptions

(a) Candidates were unable to calculate the value of r^2 correctly when answering Q4.1. Many would calculate the value of r in the following manner $\sqrt{3^2 + 4^2} = 5$ and then write the equation of the circle as $x^2 + y^2 = 5$ instead of $x^2 + y^2 = 25$.

(b) In Q4.2 a number of candidates were unable to establish the radius of the bigger circle. Instead, they incorrectly used the radius of the smaller circle in their answer: $(x + 3)^2 + (y - 4)^2 = 25$.



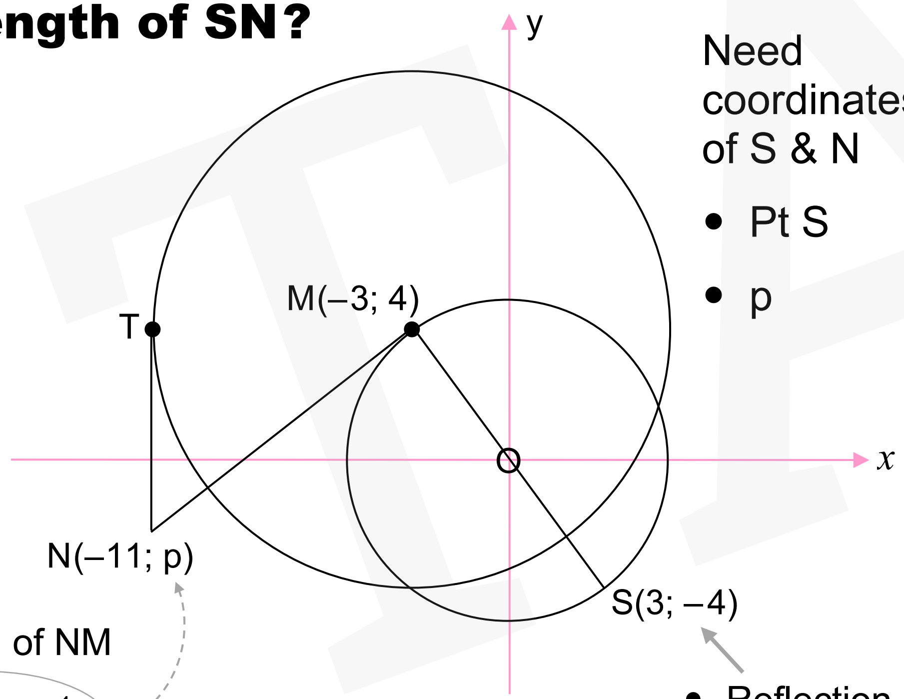
4.3 **Equation of line NM?**



A Point

Gradient

4.4 **Length of SN?**



Need coordinates of S & N

- Pt S
- p

- Reflection in the origin

- p

Equation of NM

$$y = \frac{3}{4}x + 6\frac{1}{4}$$

. . . from Q4.3

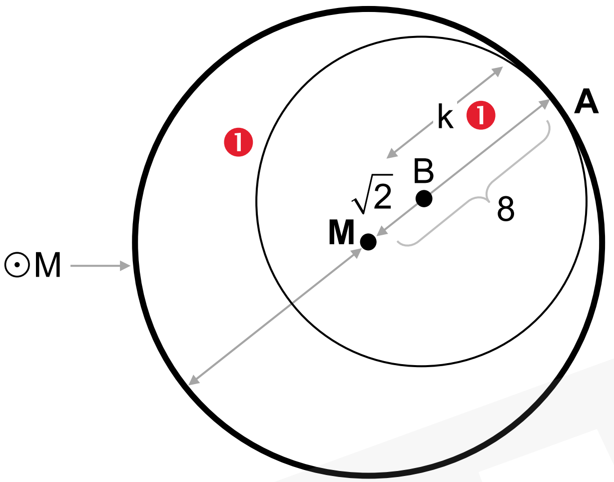
Common Errors and Misconceptions

(c) In answering Q4.3, some candidates used the gradient of OM instead of the gradient of MN to determine the equation of the line. Others could not correctly relate the gradient of OM to the gradient of MN.

(d) Candidates were unable to establish the value of p or the coordinates of S. Instead they **incorrectly made assumptions** about these values when answering Q4.4. Some candidates wrote down a positive value for p despite the point being in the third quadrant. Others incorrectly assumed that the coordinates of S were (–3 ; 4).

4.5 **Value(s) of k**

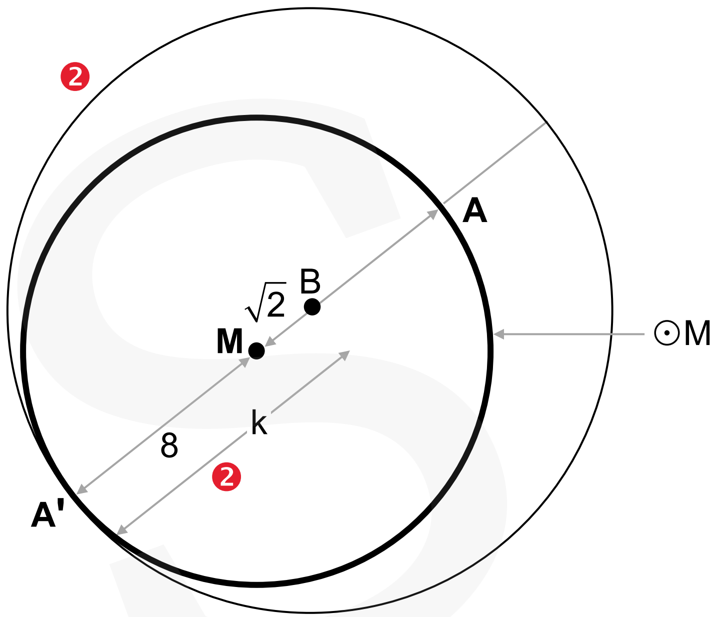
⊙B touches ⊙M at A



Whatever,

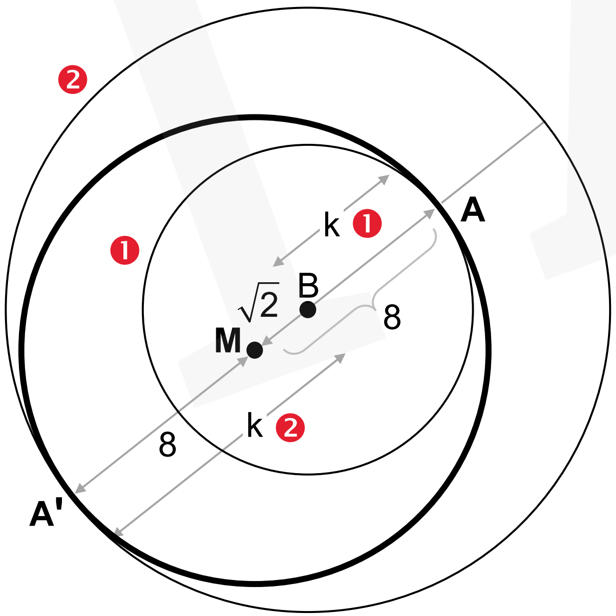
you need MB!

⊙B touches ⊙M at A'



Combined

❶ & ❷ :

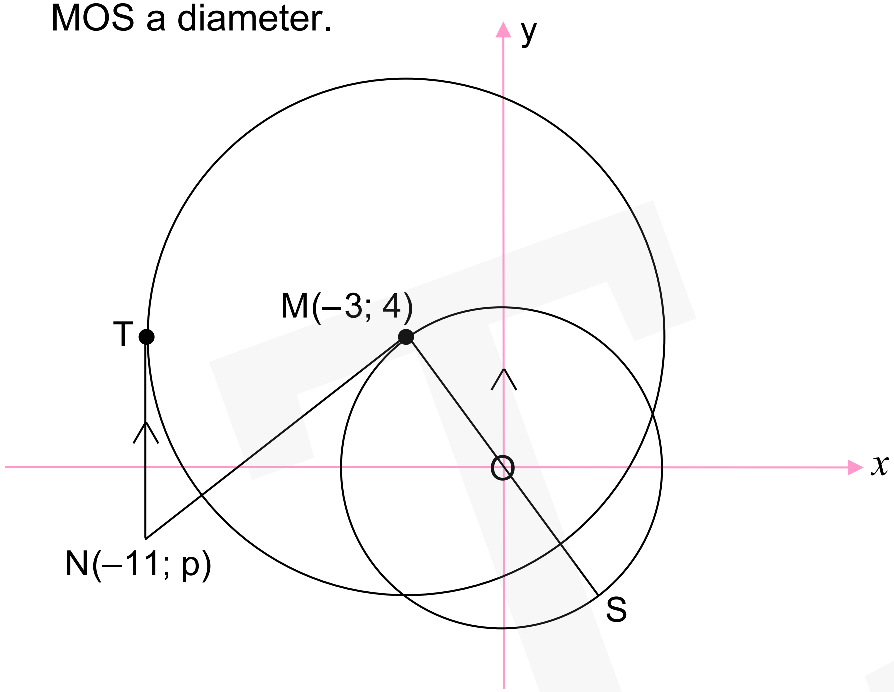


Common Errors & Misconceptions

(e) Some candidates were **unable to interpret Q4.5 correctly**. They **incorrectly assumed** that the centres of the circles were points O and M and hence calculated the length of OM unnecessarily. Many candidates could calculate the distance between the centres, B and M, to be $\sqrt{2}$ but they were unable to use this information to calculate the values of k.

QUESTION 4

- M(–3; 4) is the centre of the large circle and a point on the small circle having centre O(0; 0).
- From N(–11; p), a tangent is drawn to touch the large circle at T with NT parallel to the y-axis.
- NM is tangent to the smaller circle at M with MOS a diameter.



4.1 Determine the equation of the small circle. (2)

4.2 Determine the equation of the circle centred at M in the form $(x - a)^2 + (y - b)^2 = r^2$ (3)

4.3 Determine the equation of NM in the form $y = mx + c$. (4)

MEMO

4.1 **Small ⊙ :** radius OM = 5 units

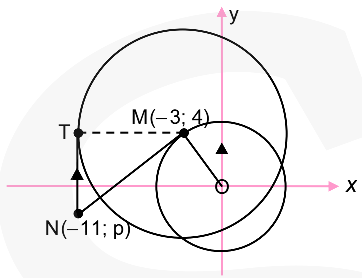
$\therefore$ **Eqn.:** $\mathbf{x^2 + y^2 = 25}$ ✓

4.2 TM is a radius of **⊙M**

& $TM = x_M - x_T$

$= -3 - (-11)$

$= 8$ units



$\therefore$ **The eqn. of ⊙M is:**

$\mathbf{(x + 3)^2 + (y - 4)^2 = 8^2}$ ✓

4.3 NM ⊥ OM & $m_{OM} = -\frac{4}{3}$

$\therefore m_{NM} = +\frac{3}{4}$ . . . *tangent ⊥ radius*

Eqn. of NM: Substitute $m = \frac{3}{4}$ & (–3; 4) into

$y = mx + c$:

$\therefore 4 = \frac{3}{4}(-3) + c$

$\therefore 4 = -2\frac{1}{4} + c$

$\therefore c = 6\frac{1}{4}$

OR: $\mathbf{y - y_1 = m(x - x_1)}$:

$\therefore y - 4 = \frac{3}{4}(x + 3)$

$\therefore y = \frac{3}{4}x + \frac{9}{4} + 4$

$\therefore$ Eqn. of NM: $\mathbf{y = \frac{3}{4}x + 6\frac{1}{4}}$ ✓

QUESTION 4 (cont.)

4.4 Calculate the length of SN. (5)

4.5 If another circle with centre B(–2; 5) and radius k touches the circle centred at M, determine the value(s) of k, correct to ONE decimal place. (5)

MEMO

4.4 Point S is (3; –4) . . . *reflection of M in the origin*

& $y_N = \frac{3}{4}(-11) + 6\frac{1}{4}$ . . . *substituting* $x_N = -11$

$= -2$

$\therefore$ Point N is (–11; –2)

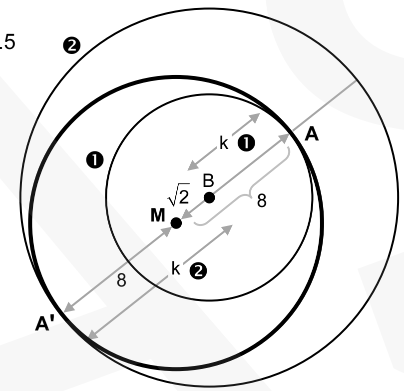
$\therefore SN^2 = (-11-3)^2 + (-2+4)^2$

$= 196 + 4$

$= 200$

$\therefore SN = \sqrt{200} \approx$ **14,14 units** ✓

4.5



The radius of ⊙**M** = 8 units

The radius of ⊙**B** = k units

BUT, there are 2 possibilities for ⊙B & $\therefore$ 2 possible values of k

Either ❶: k = **AB** = 8 – MB . . . *⊙B touches ⊙M at A*

or ❷: k = **A'B** = 8 + MB . . . *⊙B touches ⊙M at A'*

$MB^2 = (-2+3)^2 + (5-4)^2$

$= 1^2 + 1^2$

$= 2$

$\therefore MB = \sqrt{2}$ (= 1,41)

$\therefore k = 8 - \sqrt{2}$ or $8 + \sqrt{2}$

$\therefore$ **k ≈ 6,6 or 9,4** ✓ . . . *correct to ONE decimal place (as requested)*

Q4: General Suggestions for Improvement

(a) Teachers should encourage learners to **ANALYSE THE DIAGRAM** before attempting any questions. They must first **WRITE DOWN ANY GIVEN INFORMATION ON THE DIAGRAM** and **THEN MAKE DEDUCTIONS** from the given information.

(b) Teachers need to revise the concept of perpendicular lines and gradients, particularly that the tangent is perpendicular to the radius at the point of contact.

(c) Teachers should revise the work done in earlier grades.

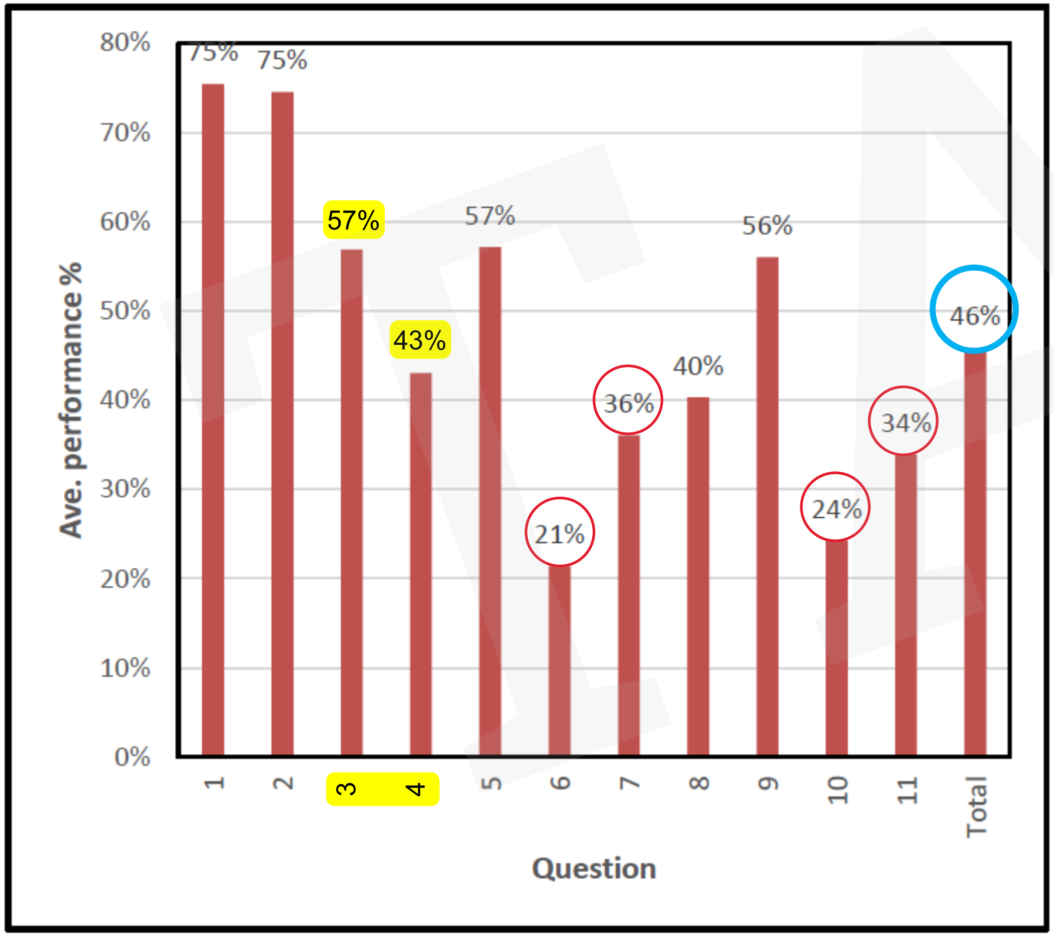
(d) Learners should be reminded to refer to the information sheet for the relevant formula.

(e) Although learners are taught how to determine the equation of a straight line from Grade 9, they should be reminded that the minimum requirements to determine the equation of a straight line are the gradient of the line and the coordinates of one point through which the line passes.

(f) Teachers should ensure that they expose learners to assessments that integrate Analytical Geometry and Euclidean Geometry. Learners must also be exposed to higher-order questions in class and in school-based assessment tasks.

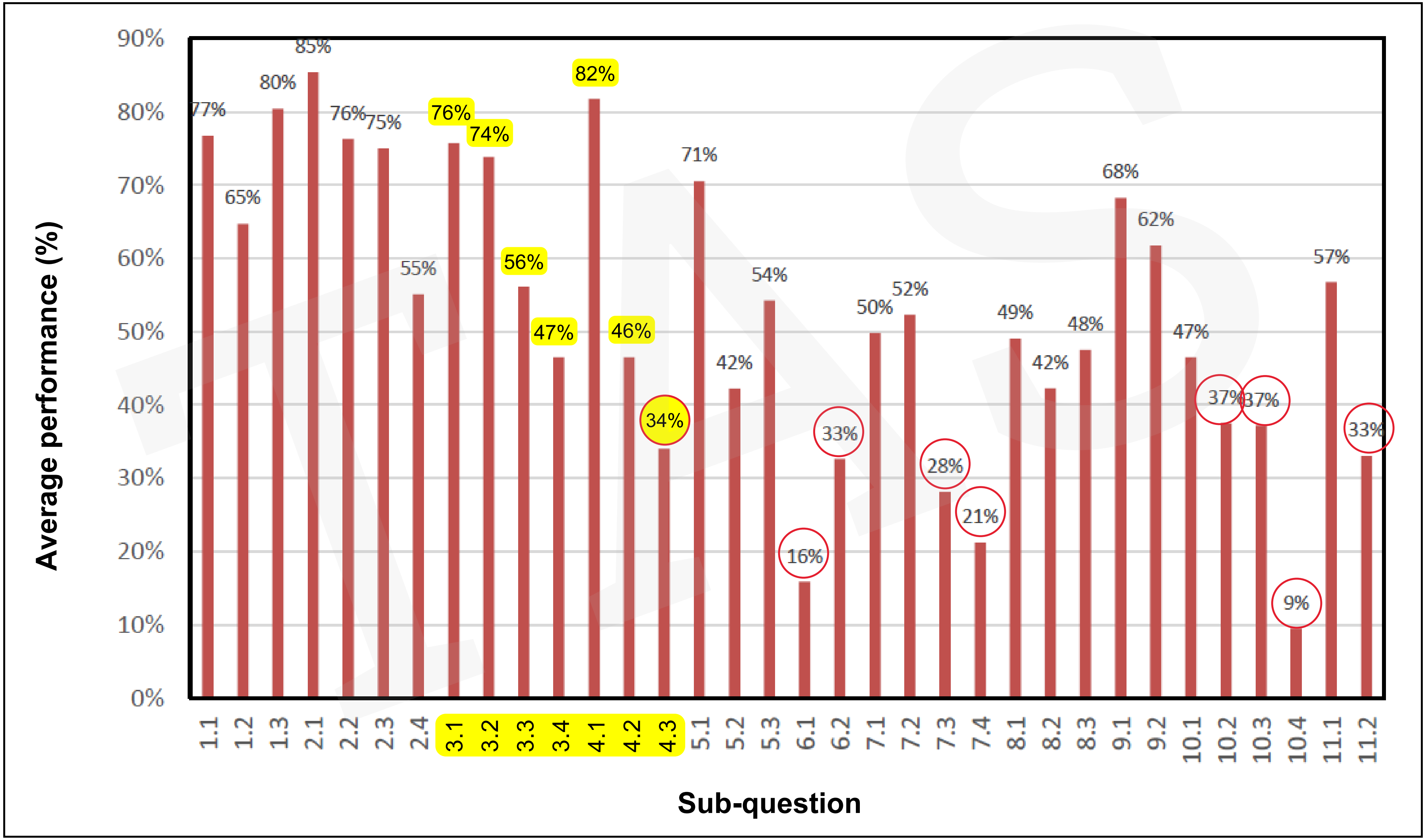
2021: Paper 2

Average % performance per question



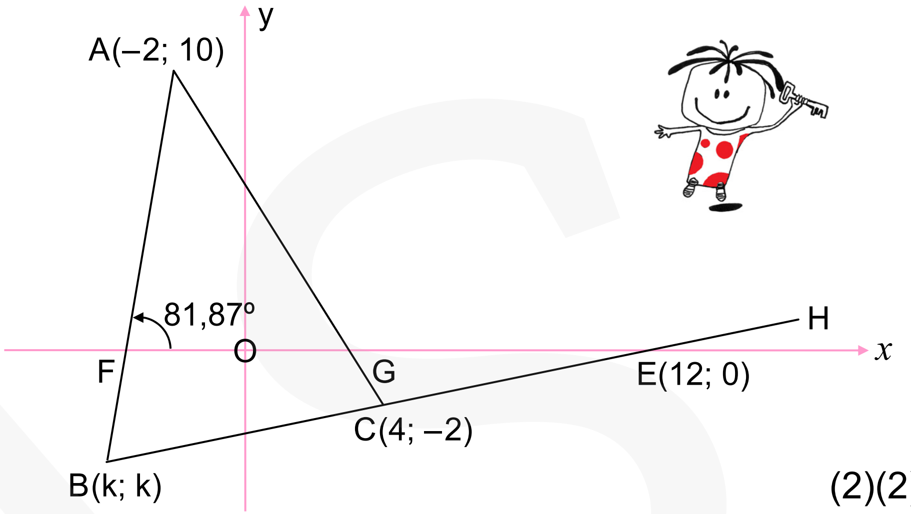
Q1	Data Handling
Q2	Data Handling
Q3	Analytical Geometry
Q4	Analytical Geometry
Q5	Trigonometry
Q6	Trigonometry
Q7	Trigonometry
Q8	Trigonometry
Q9	Euclidean Geometry
Q10	Euclidean Geometry
Q11	Euclidean Geometry

and, per sub-question



GR 12 NAT NOV 2021 – ANALYTICAL GEOMETRY

QUESTION 3 57%



- In the diagram, A(–2; 10), B(k; k) and C(4; –2) are the vertices of $\triangle ABC$.
- Line BC is produced to H and cuts the x-axis at E(12; 0).
- AB and AC intersects the x-axis at F and G respectively.
- The angle of inclination of line AB is 81,87°.

76% 3.1 Calculate the gradient of:

3.1.1 BE   3.1.2 AB (2)(2)

74% 3.2 Determine the equation of BE in the form $y = mx + c$. (2)

56% 3.3 Calculate the:

3.3.1 Coordinates of B, where $k < 0$.   3.3.2 Size of $\hat{A}$. (2)(4)

3.3.3 Coordinates of the point of intersection of the diagonals of parallelogram ACES, where S is a point in the first quadrant. (2)

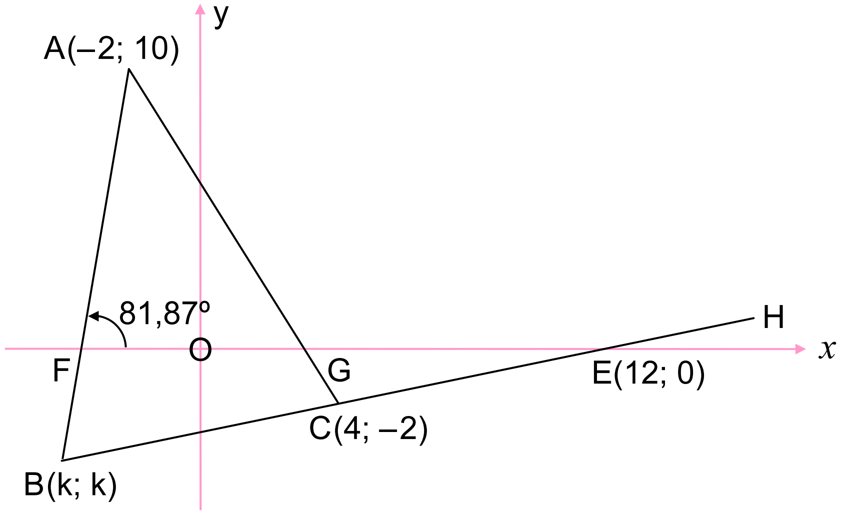
47% 3.4 Another point T(p; p), where $p > 0$, is plotted such that $ET = BE = 4\sqrt{17}$ units.

3.4.1 Calculate the coordinates of T. (5)

3.4.2 Determine the equation of the:

(a) Circle with centre at E and passing through B and T in the form $(x - a)^2 + (y - b)^2 = r^2$. (2)

(b) Tangent to the circle at point B(k; k). (3) **[24]**



3.1 Calculate the gradient of:

3.1.1 BE (2)

MEMO

3.1.1 $m_{BE} = \frac{0-(-2)}{12-4} = \frac{2}{8} = \mathbf{\frac{1}{4}}$ ✓

3.1.2 AB (2)

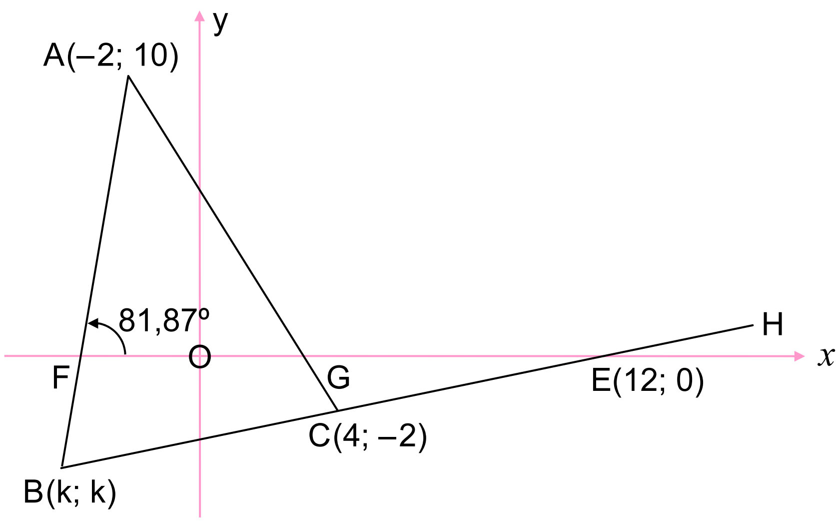
MEMO

3.1.2 $m_{AB} = \tan 81{,}87° = \mathbf{7}$ ✓

Common Errors and Misconceptions

(a) Many candidates failed to recognise that B, C and E were collinear points and hence, when answering Q3.1, failed to realise that $m_{BE} = m_{CE}$. Some substituted the coordinates of B into the gradient formula and ended up with an answer as an expression containing k. Some candidates still write the gradient formula incorrectly, despite it being given in the information sheet. Some candidates incorrectly used BE as the notation for the gradient of BE instead of m_{BE}.

(b) In Q3.1.2 many candidates used $\tan^{-1}(81{,}87°)$ to calculate the gradient of AB instead of $\tan 81{,}87°$. This shows that candidates were confused between gradient and angle of inclination. Some candidates used the answers for k obtained later in Q3.3.1 to calculate the gradient of AB. This was not accepted as the calculations for k were not done prior to answering Q3.1.2.



3.2 Determine the equation of BE in the form y = mx + c. (2)

MEMO

3.2 Subst. $m = \frac{1}{4}$ & pt. (12; 0) in

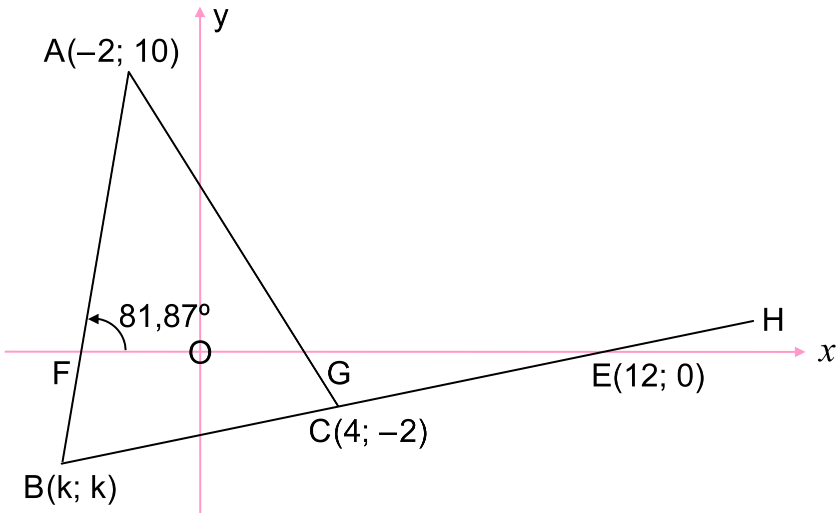
$$y - y_1 = m(x - x_1)$$

$$\therefore\ y - 0 = \frac{1}{4}(x - 12)$$

$\therefore$ **Eqn of BE:** $\therefore\ \mathbf{y = \frac{1}{4}x - 3}$ ✓

Common Errors and Misconceptions

(c) When answering Q3.2, some candidates calculated the y-intercept of BE correctly but failed to write down the equation of BE. Their answer was incomplete and they were not awarded the mark for the equation of BE.



3.3 Calculate the:

3.3.1 Coordinates of B, where $k < 0$. (2)

MEMO

3.3.1 Pt B(k; k) on BE: $\therefore\ k = \frac{1}{4}k - 3$

$(\times 4)\quad \therefore\ 4k = k - 12$

$\therefore\ 3k = -12$

$\therefore\ k = -4$

$\therefore$ **B(–4; –4)** ✓

OR: Pt of intersection of $y = x$ & $y = \frac{1}{4}x - 3$

$\therefore\ x = \frac{1}{4}x - 3$

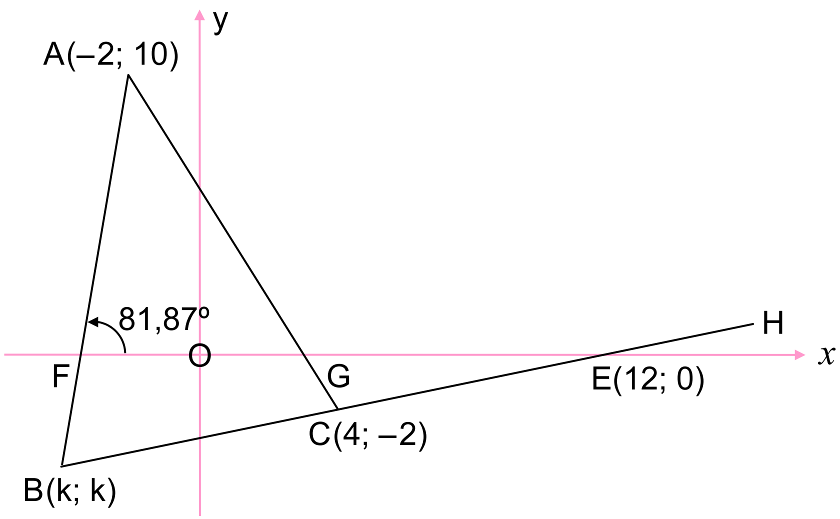
$(\times 4)\quad \therefore\ 4x = x - 12$

$\therefore\ 3x = -12$

$\therefore\ x = -4 \qquad \therefore$ **B(–4; –4)** ✓

Common Errors and Misconceptions

(d) In Q3.3.1 many candidates incorrectly **assumed that C was the midpoint of BE**. This information was not given and these candidates were not awarded any marks for their efforts.



3.3 Calculate the:

3.3.2 Size of $\hat{A}$. (4)

MEMO

3.3.2 $m_{AC} = \frac{10-(-2)}{-2-4} = \frac{12}{-6} = -2$

$\therefore\ A\hat{G}E = 180^\circ - 63{,}43^\circ$

$= 116{,}57^\circ$

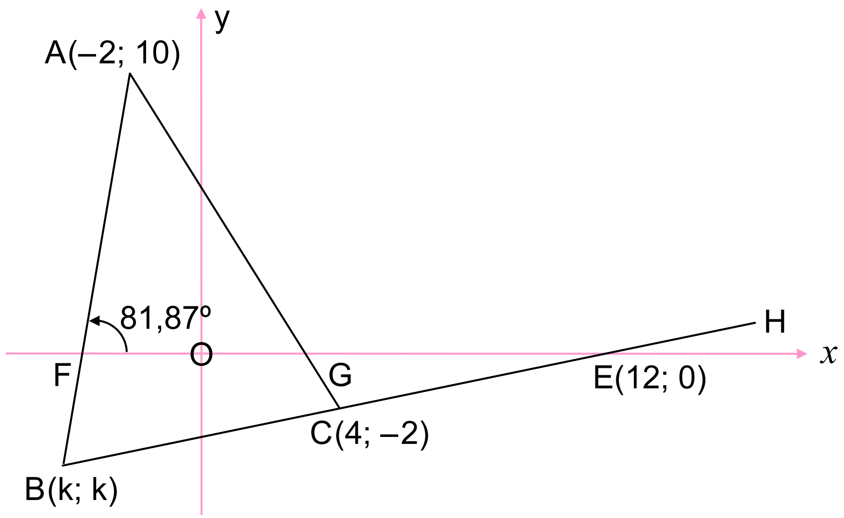
$\therefore\ \hat{A} = A\hat{G}E - A\hat{F}E$ ... *ext. ∠ of Δ*

$= 116{,}57^\circ - 81{,}87^\circ$

$= \mathbf{34{,}70^\circ}$ ✓

Common Errors and Misconceptions

(e) Many candidates correctly calculated the gradient of AC as –2 but did not realise that this implied that $A\hat{G}E$ was obtuse. Although some candidates were able to calculate $A\hat{G}E$ and $A\hat{F}G$ correctly, they were unable to relate these angles to $\hat{A}$ when answering Q3.3.2.



3.3 Calculate the:

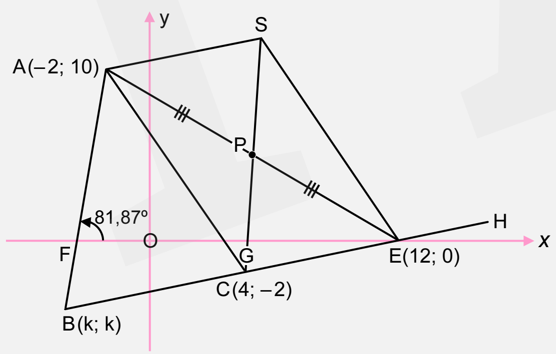
3.3.3 Coordinates of the point of intersection of the diagonals of parallelogram ACES, where S is a point in the first quadrant. (2)

MEMO

3.3.3 Pt of intersection of diagonals, say pt P is the midpt of AE

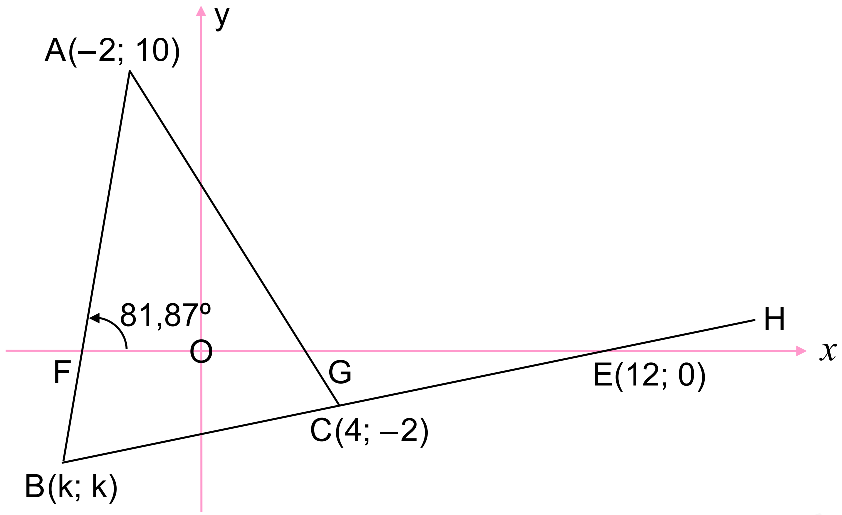
$\therefore\ P\left(\frac{-2+12}{2};\ \frac{10+0}{2}\right)$

$\therefore$ **P(5; 5)** ✓



Common Errors and Misconceptions

(f) The point S was not shown on the sketch. Many candidates failed to attempt this question because they lacked the visual skills to correctly place point S in the first quadrant.



3.4 Another point T(p; p), where p > 0, is plotted such that ET = BE = $4\sqrt{17}$ units.

3.4.1 Calculate the coordinates of T. (5)

MEMO

3.4.1 ET = BE = $4\sqrt{17}$

But $ET^2 = (p - 12)^2 + (p - 0)^2$

$= p^2 - 24p + 144 + p^2$

$= 2p^2 - 24p + 144$

$\therefore\ 2p^2 - 24p + 144 = 4^2 \times 17$

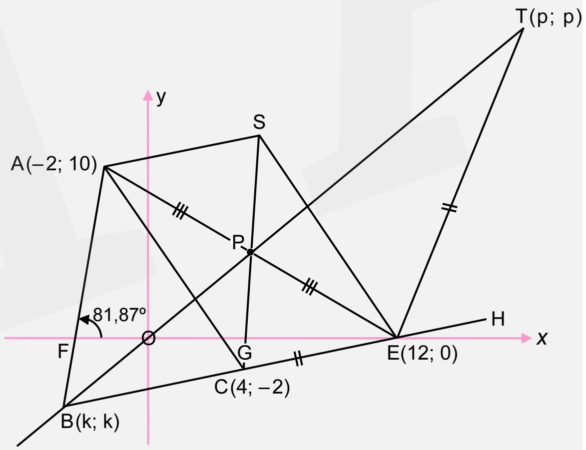
$\therefore\ 2p^2 - 24p - 128 = 0$

(÷2) $\therefore\ p^2 - 12p - 64 = 0$

$\therefore\ (p - 16)(p + 4) = 0$

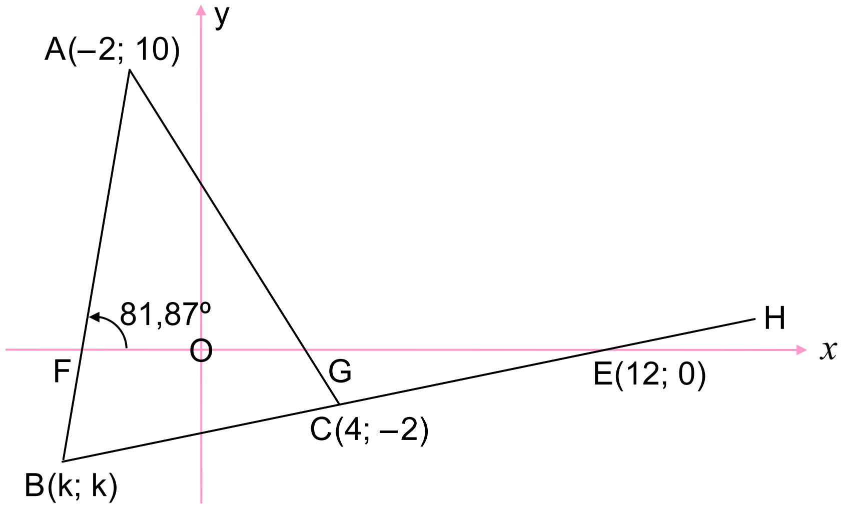
$\therefore\ p = 16 \quad \ldots\ p \neq -4 \quad \because\ p > 0$

$\therefore$ **T(16; 16)**



Common Errors and Misconceptions

(g) Many candidates did not substitute p for both x and y in the equation for BE. Consequently, they were unable to determine the coordinates of T as the equation contained two variables. Again, candidates lacked the visual skills to correctly place T in the first quadrant.



3.4.2 Determine the equation of the:

(a) Circle with centre at E and passing through B and T in the form $(x - a)^2 + (y - b)^2 = r^2$. (2)

MEMO

3.4.2 (a) Radius ET = BE = $4\sqrt{17}$

$\therefore\ r^2 = 16 \times 17 = 272$

& Centre E(12; 0)

$\therefore$ Eqn of ⊙E: $\mathbf{(x - 12)^2 + y^2 = 272}$ ✓

(b) Tangent to the circle at point B(k; k). (3)

MEMO

(b) $m_{BE} = \frac{1}{4}$ . . . *in 3.1.1*

$\therefore$ Gradient of tangent at B = –4

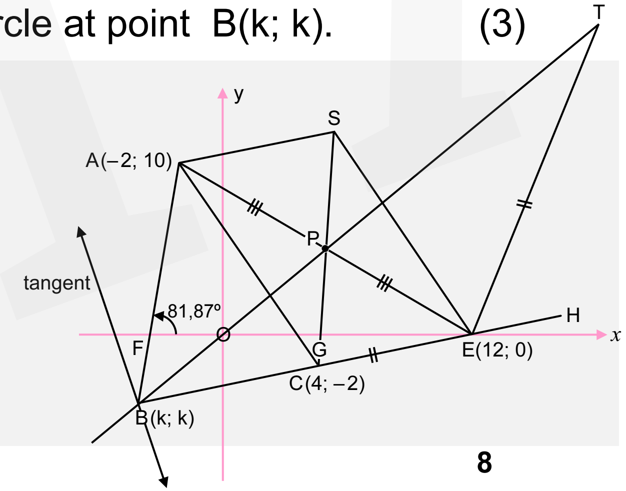
& Pt B(–4; –4) . . . *in 3.3.1*

Subst. in $y - y_1 = m(x - x_1)$

$\therefore\ y + 4 = -4(x + 4)$

$\therefore\ y = -4x - 16 - 4$

$\therefore\ \mathbf{y = -4x - 20}$ ✓



Common Errors and Misconceptions

(h) The centre of the circle was given. Many candidates were able to use this information to write down the LHS of the equation correctly in Q3.4.2(a). However, they did not realise that BE was the radius of the required circle.

(i) Determining the equation of a tangent to a circle at the point of contact is a familiar question. However, many candidates lacked the visual skills to see that there was a circle passing through B and that they had to calculate the equation of the tangent passing through B.

Question 3: Suggestions for Improvement

(a) If learners are not sure, they should consult the information sheet for the correct formula.

(b) **Substitution** into the formula remains **a problem**. Learners should first write down the coordinates and then substitute them into the formula.

(c) Teachers should request learners to label the coordinates as $(x_1\ ;\ y_1)$ and $(x_2\ ;\ y_2)$ on the diagram. This should prevent learners from making mistakes when substituting the coordinates into a formula. The order of substitution must be consistent, especially when using the gradient formula.

(d) Emphasise to learners that it is not acceptable to make any assumptions, e.g. that a certain point is the midpoint of a line. Even if it looks as if the point is the midpoint, it may not just be assumed and used. These need to be proved first before the results can be used in an answer.

(e) Teachers should encourage learners to write down the values that they have already calculated (lengths, angles and gradients) on the diagram. This will assist learners when answering follow-up questions.

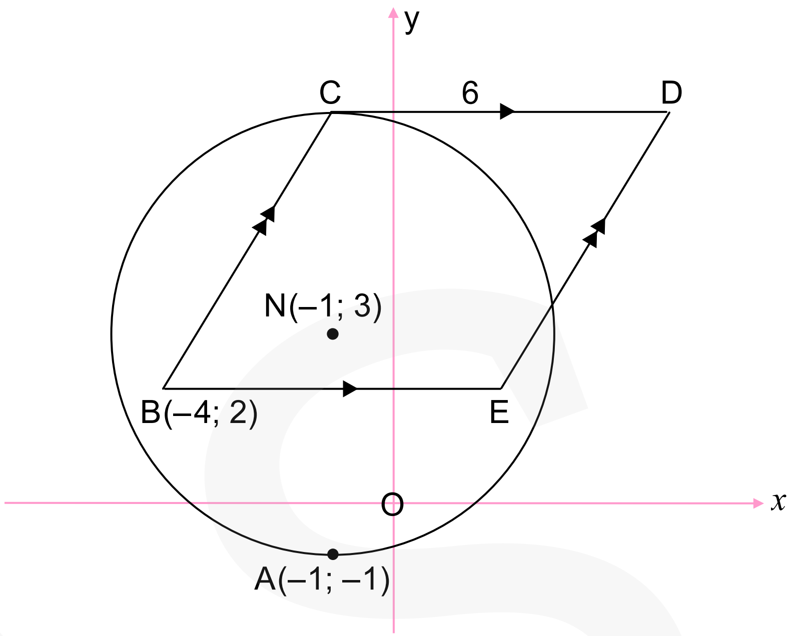
(f) To answer questions in analytical geometry well, learners should master the properties of **quadrilaterals and triangles**. Constant revision of Analytical Geometry concepts taught in Grades 10 and 11 is essential, as much of the Grade 12 work revolves around these concepts.

(g) The different topics in Mathematics should be integrated. Learners must be able to establish the connection between Euclidean Geometry and Analytical Geometry.

(h) Learners have difficulty in visualising the figures and points not shown on a sketch. Teachers need to inculcate the skill of visualising and drawing the given information.

QUESTION 4 43%

- In the diagram, the circle centred at N(–1; 3) passes through A(–1; –1) and C.
- B(–4; 2), C, D and E are joined to form a parallelogram such that BE is parallel to the x-axis.
- CD is a tangent to the circle at C and CD = 6 units.



82% 4.1 Write down the length of the radius of the circle. (1)

46% 4.2 Calculate the:

4.2.1 Coordinates of C. (2)

4.2.2 Coordinates of D. (2)

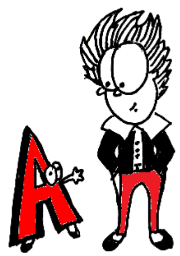
4.2.3 Area of $\triangle$BCD. (3)

34% 4.3 The circle, centred at N, is reflected about the line y = x. M is the centre of the new circle which is formed. The two circles intersect at A and F.

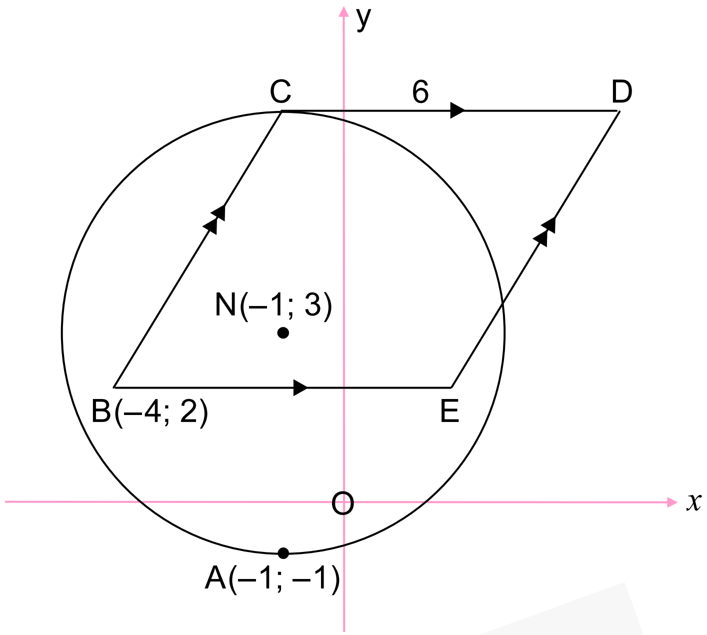
Calculate the:

4.3.1 Length of NM. (3)

4.3.2 Midpoint of AF. (4)



[15]



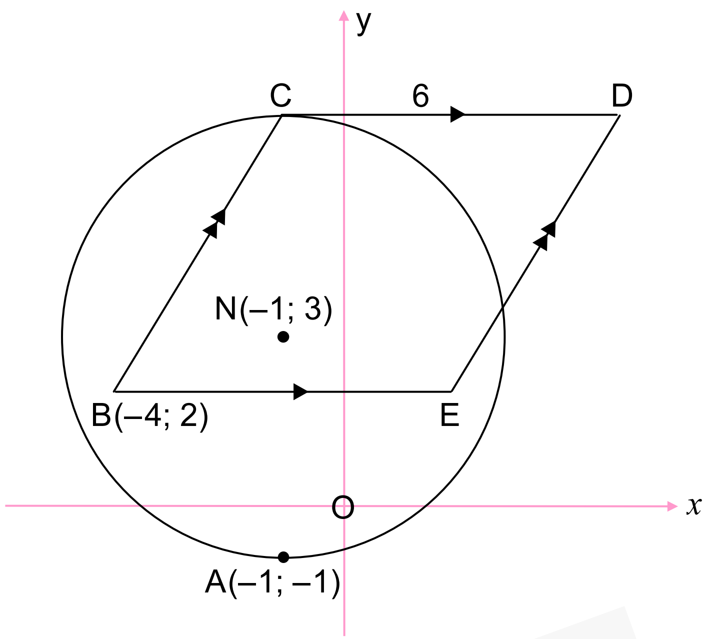
4.1 Write down the length of the radius of the circle. (1)

MEMO

4.1 The length of the radius = NA = 3 – (–1)
= 4 units

Common Errors and Misconceptions

(a) Many candidates used the distance formula to calculate the radius. This was not necessary since the centre and the point A have the same x-coordinate, and all that was required was to subtract the y-coordinates of these two points.



4.2 Calculate the:

4.2.1 Coordinates of C. (2)

MEMO

4.2.1 At C: $x_C = -1$

& $y_C = 3 + 4 = 7$ units

$\therefore$ **C(–1; 7)** ◂

4.2.2 Coordinates of D. (2)

MEMO

4.2.2 At D: $y_D = y_C = 7$ . . . *CD || BE || x-axis*

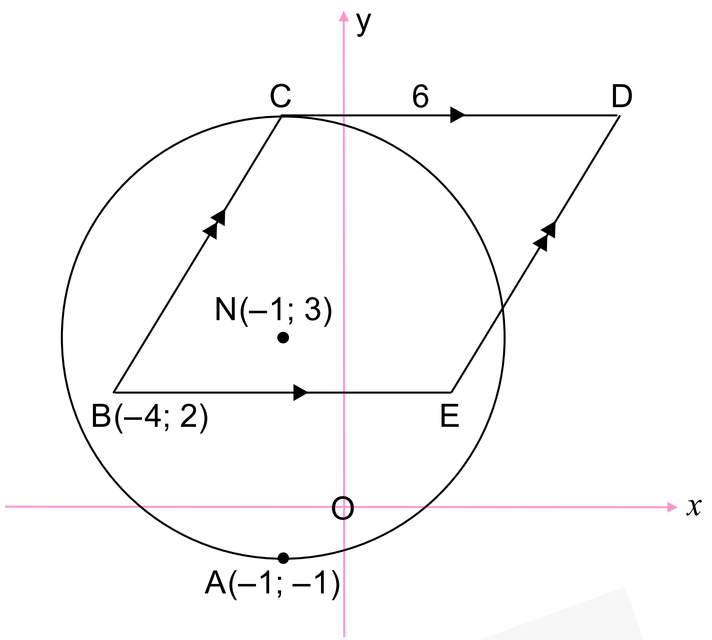
& $x_D = x_C + 6 = -1 + 6 = 5$

$\therefore$ **D(5; 7)** ◂

Common Errors and Misconceptions

(b) In Q4.2.1 a number of candidates were unable to establish that BE and CD were both parallel to the x-axis and therefore these lines were perpendicular to CN, the radius of the circle. Consequently, they were unable to determine the coordinates of C.

(c) Candidates were unable to make the link between the coordinates of C and the distance of 6 units in order to calculate the coordinates of D in Q4.2.2.



4.2.3 Area of $\triangle BCD$. (3)

MEMO

4.2.3 Area of $\triangle BCD$ $= \frac{1}{2}CD \times$ height (C to BE) . . . $\mathbf{A = \frac{1}{2}bh}$

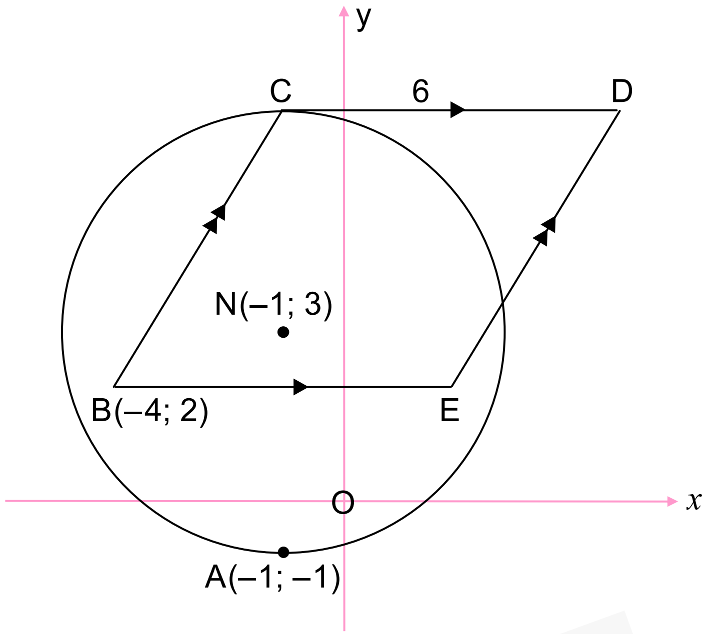
$= \frac{1}{2}(6) \times (y_C - y_B)$

$= 3 \times (7 - 2)$

$=$ **15 units^2**

Common Errors and Misconceptions

(d) When answering Q4.2.3, many candidates had difficulty in identifying **the height of $\triangle$BCD**. A number of candidates **used BD as the base** but were unable to calculate the height of the triangle.



4.3 The circle, centred at N, is reflected about the line y = x. M is the centre of the new circle which is formed. The two circles intersect at A and F.

Calculate the:

4.3.1 Length of NM. (3)

MEMO

4.3.1 Coordinates of M(3; –1) . . . *reflection in the line y = x*

$\therefore$ $NM^2 = (3 + 1)^2 + (-1 - 3)^2$

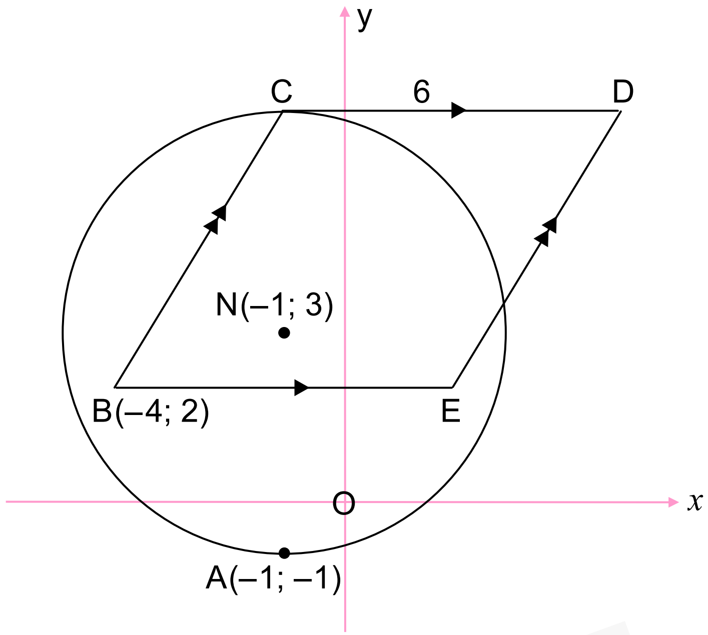
$= 16 + 16$

$= 32$

$\therefore$ **NM** $= \sqrt{32} = 4\sqrt{2} \approx$ **5,66 units**

Common Errors and Misconceptions

(e) In answering Q4.3.1, some candidates could not recall the rule for reflecting a point about the line y = x. Many just swopped the signs without interchanging the x- and y-coordinates. Their coordinates of M were (1; –3), which was incorrect.



Calculate the:

4.3.2 Midpoint of AF. (4)

Common Errors and Misconceptions

(f) Many candidates did not attempt Q4.3.2 because they could not place F on the diagram.

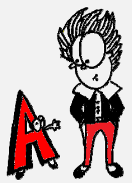
MEMO

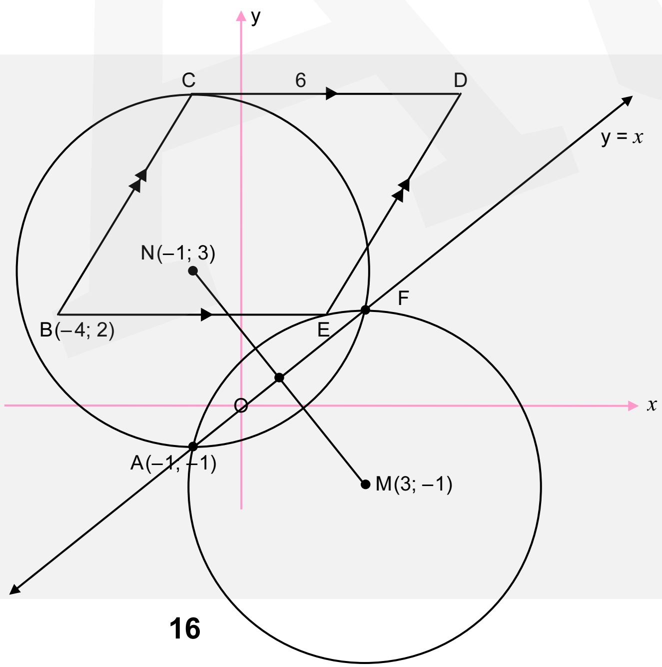
4.3.2 ANFM is a rhombus . . . *radii equal*

$\therefore$ Midpoint of AF is the midpoint of MN . . . *diags of rhombus*

$\therefore \left(\frac{-1+3}{2}; \frac{3+(-1)}{2}\right)$

$\therefore$ **(1; 1)** ✓





Question 4: Suggestions for Improvement

(a) Teachers should encourage learners to **analyse the diagram** before attempting any questions. They must first write down any given information on the diagram and then make deductions from the given information.

(b) Teachers need to revise the concept of perpendicular lines and gradients, particularly that the tangent is perpendicular to the radius at the point of contact. Teachers should also show learners why it is sufficient to subtract the x-coordinates to calculate the distance between two points in a horizontal plane and why it is sufficient to subtract the y-coordinates to calculate the distance between two points in a vertical plane.

(c) Teachers should revise the **work done in earlier grades**. The properties of all the special **quadrilaterals**, e.g. the parallelogram, rhombus and square, should be taught thoroughly in earlier grades so that whenever that knowledge is needed, learners will be able to use it.

(d) Learners should be reminded to refer to the information sheet for the relevant formula.

(e) Teachers should show learners how to visualise and make rough drawings of all extra information given in Analytical Geometry questions.

(f) Teachers should show learners different orientations of the base and the perpendicular height of a triangle. This should give learners more options when calculating the area of a triangle.

(g) Teachers should ensure that they expose learners to assessments that integrate Analytical Geometry and Euclidean Geometry. Learners must also be exposed to higher-order questions in class and in school-based assessment tasks.

THE CAPS CURRICULUM: OVERVIEW OF TOPICS

Paper 2

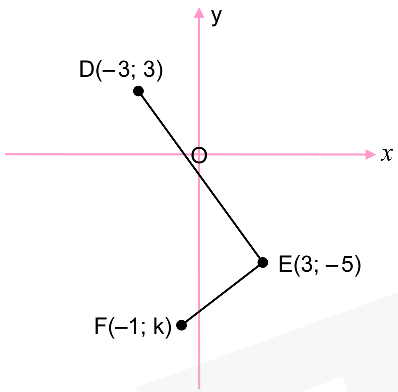
9. ANALYTICAL GEOMETRY

Grade 10	Grade 11	Grade 12
Represent geometric figures in a Cartesian co-ordinate system, and derive and apply, for any two points $(x_1 ; y_1)$ and $(x_2 ; y_2)$, a formula for calculating: • the distance between the two points; • the gradient of the line segment joining the points; • conditions for parallel and perpendicular lines; and • the co-ordinates of the mid-point of the line segment joining the points.	Use a Cartesian co-ordinate system to derive and apply: • the equation of a line through two given points; • the equation of a line through one point and parallel or perpendicular to a given line; and • the inclination of a line.	Use a two-dimensional Cartesian co-ordinate system to derive and apply: • the equation of a circle (any centre); and • the equation of a tangent to a circle at a given point on the circle.

GR 10 – 12 EXEMPLAR ANALYTICAL GEOMETRY

GRADE 10: QUESTIONS

3.1 In the diagram below, D(–3; 3), E(3; –5) and F(–1; k) are three points in the Cartesian plane.



3.1.1 Calculate the length of DE. (2)

3.1.2 Calculate the gradient of DE. (2)

3.1.3 Determine the value of k if $D\hat{E}F = 90°$. (4)

3.1.4 If k = –8, determine the coordinates of M, the midpoint of DF. (2)

3.1.5 Determine the coordinates of a point G such that the **quadrilateral** DEFG is a **rectangle**. (4)

3.2 C is the point (1; –2). The point D lies in the second quadrant and has coordinates $(x; 5)$.

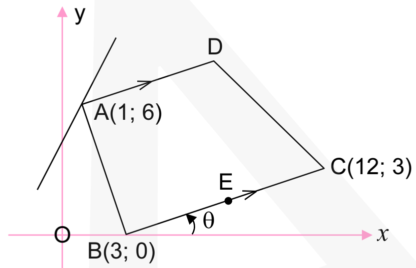
If the length of CD is $\sqrt{53}$ units, calculate the value of x. (4) [18]

GRADE 11: QUESTIONS

3. A(1; 6), B(3; 0), C(12; 3) and D are the vertices of a **trapezium** with AD || BC.

E is the midpoint of BC.

The angle of inclination of the straight line BC is θ, as shown in the diagram.



Midpoint

3.1 Calculate the coordinates of E. (2)

3.2 Determine the gradient of the line BC. (2)

3.3 Calculate the magnitude of θ. (2)

3.4 Prove that AD is perpendicular to AB. (3)

3.5 A straight line passing through vertex A does not pass through any of the sides of the **trapezium**.

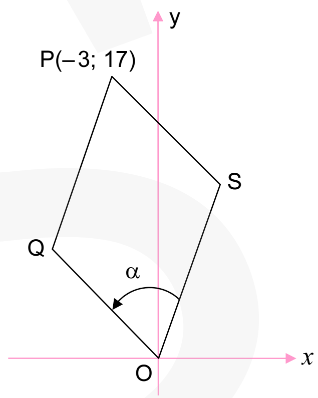
This line makes an angle of 45° with side AD of the trapezium. Determine the equation of this straight line. (5)

[14]

4. In the diagram below, P(–3; 17), Q, O and S are the vertices of a **parallelogram**.

The sides OS and OQ are defined by the equations $y = 6x$ and $y = -x$ respectively.

$Q\hat{O}S = \alpha$.



4.1 Determine the equation of QP in the form $y = mx + c$. (3)

Point of Intersection

4.2 Hence, determine the coordinates of Q. (4)

4.3 Calculate the length of OQ. Leave your answer in simplified surd form. (2)

4.4 Calculate the size of α. (3)

4.5 If OS = $\sqrt{148}$ units, calculate the length of QS. (3)

[15]

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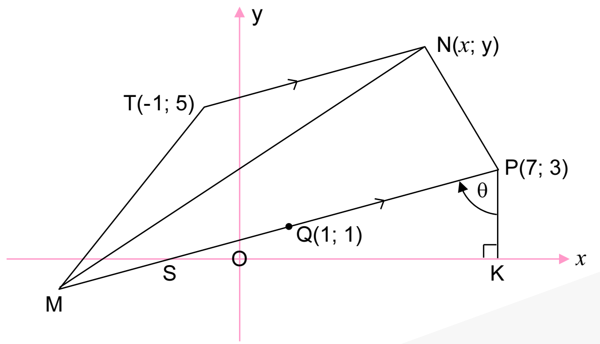
GRADE 12: QUESTIONS

3. In the diagram below, M, T(–1; 5), N(x; y) and P(7; 3) are vertices of **trapezium** MTNP having TN || MP. Q(1; 1) is the midpoint of MP. PK is a vertical line and $S\hat{P}K = \theta$. The equation of NP is $y = -2x + 17$.



3.1 Write down the coordinates of K. (1)

3.2 Determine the coordinates of M. (2)

3.3 Determine the gradient of PM. (2)

3.4 Calculate the size of θ. (3)

3.5 Hence, or otherwise, determine the length of PS. (3)

Point of Intersection

3.6 Determine the coordinates of N. (5)

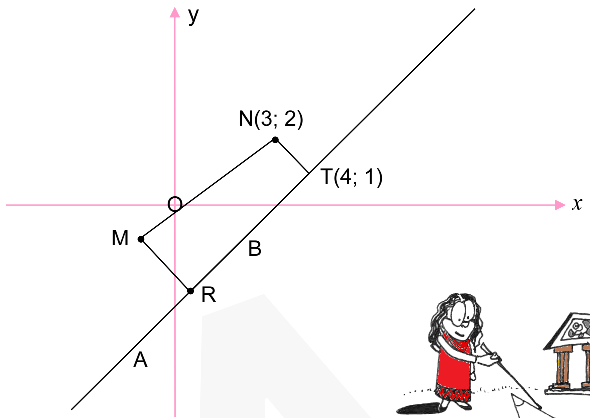
3.7 If A(a; 5) lies in the Cartesian plane:

3.7.1 Write down the equation of the straight line representing the possible positions of A. (1)

3.7.2 Hence, or otherwise, calculate the value(s) of a for which $T\hat{A}Q = 45°$. (5)

[22]

4. In the diagram below, the equation of the circle having centre M is $(x + 1)^2 + (y + 1)^2 = 9$. R is a point on chord AB such that MR bisects AB. ABT is a tangent to the circle having centre N(3; 2) at point T(4; 1).



Centre of ⊙

4.1 Write down the coordinates of M. (1)

4.2 Determine the equation of AT in the form $y = mx + c$. (5)

4.3 If it is further given that $MR = \frac{\sqrt{10}}{2}$ units, calculate the length of AB. Leave your answer in simplest surd form. (4)

4.4 Calculate the length of MN. (2)

4.5 Another circle having centre N touches the circle having centre M at point K. Determine the equation of the new circle. Write your answer in the form $x^2 + y^2 + Cx + Dy + E = 0$ (3) [15]

Touching ⊙s!

General form

USE THE DIAGRAM!

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3.1.1 $DE^2 = (3 + 3)^2 + (-5 - 3)^2$

$= 36 + 64$

$= 100$

$\therefore$ DE = 10 units ◄

3.1.2 Gradient of DE,

$m_{DE} = \dfrac{-5 - 3}{3 + 3} = \dfrac{-8}{6} = -\dfrac{4}{3}$ ◄

3.1.3 $m_{EF} = \dfrac{k + 5}{-1 - 3} = \dfrac{k + 5}{-4}$

$D\hat{E}F = 90°$ ➡ $m_{EF} = +\dfrac{3}{4}$ . . . *EF ⊥ DE*

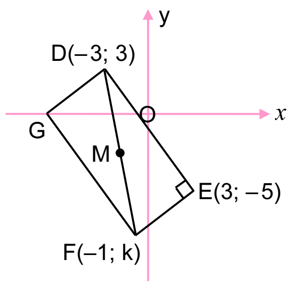
$\therefore \dfrac{k + 5}{-4} = \dfrac{3}{4}$

$\times(-4)$ $\therefore k + 5 = -3$

$\therefore k = -8$ ◄

3.1.4 $M\left(\dfrac{-3 + (-1)}{2} ; \dfrac{3 + (-8)}{2}\right)$

$\therefore M\left(-2 ; -\dfrac{5}{2}\right)$ ◄



3.1.5

DEFG will be a $||^m$ if M is the midpoint of EG too.

& Since $D\hat{E}F = 90°$,
DEFG will be a rectangle.
. . . *if one ∠ of a $||^m$ is a right ∠, then the $||^m$ is a rectangle.*

$\dfrac{x_G + 3}{2} = -2$ and $\dfrac{y_G + (-5)}{2} = -\dfrac{5}{2}$

$\times 2)$ $\therefore x_G + 3 = -4$ $\therefore y_G - 5 = -5$

$\therefore x_G = -7$ $\therefore y_G = 0$

$\therefore$ G(–7; 0) ◄

OR: The **translation** F to G equals that of E to D

$\therefore$ G(–1 – 6; –8 + 8)

$\therefore$ G(–7; 0) ◄

OR: The **translation** D to G equals that of E to F

$\therefore$ G(–3 – 4; 3 – 3)

$\therefore$ G(–7; 0) ◄

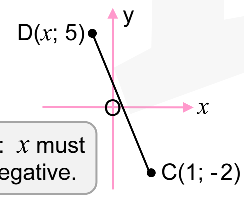
3.2 $CD^2 = (x - 1)^2 + (5 + 2)^2 = \left(\sqrt{53}\right)^2$

$\therefore (x - 1)^2 + 49 = 53$

$\therefore (x - 1)^2 = 4$

$\therefore x - 1 = \pm 2$

$\therefore x = 3$ or -1



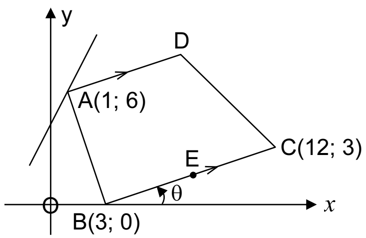
Note: x must be negative.

But $x < 0$ in the second quadrant

$\therefore x = -1$ ◄ . . . *only the neg. value of x is valid*

GRADE 11: MEMOS

3.



3.1 Point E is $\left(\frac{3+12}{2}; \frac{0+3}{2}\right)$, ... $\left(\frac{x_1+x_2}{2}; \frac{y_1+y_2}{2}\right)$

$\therefore$ **E$\left(7\frac{1}{2}; 1\frac{1}{2}\right)$** ◄

3.2 $m_{BC} = \frac{3-0}{12-3} = \frac{3}{9} = \mathbf{\frac{1}{3}}$ ◄ ... $m = \frac{y_2 - y_1}{x_2 - x_1}$

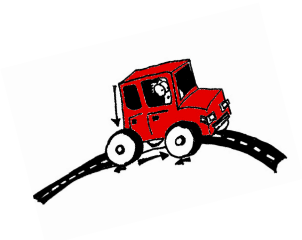
3.3 $\tan\theta = \frac{1}{3}$ ➡ $\theta \approx$ **18,43º** ◄

3.4 $m_{AD} = m_{BC} = \frac{1}{3}$ ... *AD || BC*

& $m_{AB} = \frac{0-6}{3-1} = \frac{-6}{2} = -3$

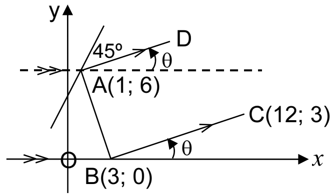
$\therefore m_{AD} \times m_{AB} = \left(\frac{1}{3}\right)(-3) = -1$

$\therefore$ **AD ⊥ AB** ◄



3.5 We need the gradient of the line.
$\therefore$ We need the ∠ of inclination.

Draw a horizontal line (|| x-axis) through point A.



The ∠ of inclination of the line is θ + 45º, i.e. 63,43º.

$\therefore$ The gradient of the line is tan 63,43º ≈ 2

AD || BC
∴ They have equal ∠ˢ of inclination.

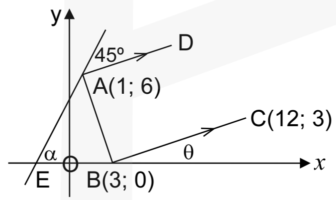
The equation: **y - y₁ = m(x - x₁)**

Substitute pt. A(1; 6): $y - 6 = 2(x-1)$

$\therefore$ **y = 2x + 4** ◄

or **y = mx + c**
$\therefore$ 6 = (2)(1) + c
$\therefore$ 4 = c
$\therefore$ Eqn.: **y = 2x + 4** ◄

OR: Extend the line to cut the x-axis (at E)



AB̂C = 90º ... *co-int. ∠ˢ ; AD || BC*

$\therefore$ AB̂X = 90º + 18,43º
= 108,43º

$\therefore$ EÂB = 45º ... *∠ˢ on a straight line*

$\therefore$ α = 108,43º – 45º ... *ext. ∠ of ΔAEB*
= 63,43º

m_{EA} = tan 63,43º ≈ 2, etc.

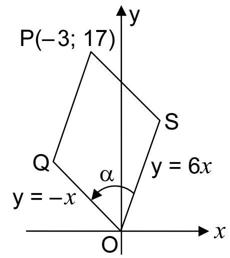
4.1 $m_{QP} = m_{OS} = 6$ ... *QP || OS in ||ᵐ*

& Substitute point P(–3; 17):

$y - 17 = 6(x+3)$
$\therefore$ **y = 6x + 35** ◄

OR: 17 = (6)(–3) + c
$\therefore$ 35 = c
$\therefore$ Eqn.: **y = 6x + 35** ◄

4.2 At Q: y = 6x + 35 and y = –x

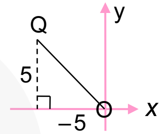


$\therefore$ 6x + 35 = –x
$\therefore$ 7x = –35
$\therefore$ x = –5
& $\therefore$ y = 5

$\therefore$ **Q(–5; 5)** ◄

4.3 $OQ^2 = 5^2 + 5^2$ ... *Thm. of Pythag.*
= 50

$\therefore$ OQ = $\sqrt{50}$
= **5$\sqrt{2}$ units** ◄



$\sqrt{50} = \sqrt{25 \times 2} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$

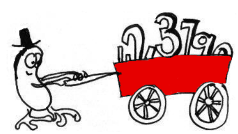
4.4 tan QÔX = –1 ... $m_{OQ} = -1$
$\therefore$ QÔX = 135º

tan SÔX = 6 ... $m_{OS} = 6$
$\therefore$ SÔX = 80,54º

$\therefore$ α = 135º – 80,54º
= **54,46º** ◄

4.5 In ΔQOS: $QS^2 = OQ^2 + OS^2 - 2OQ.OS \cos\alpha$
$= 50 + 148 - 2\sqrt{50}\sqrt{148}.\cos 54{,}46º$
= 97,994...

$\therefore$ **QS ≃ 9,90 units** ◄



GRADE 12: MEMOS

3.1 **K(7; 0)** ◄

3.2 **M(-5; –1)** ◄ . . . *Q midpoint of MP*

3.3 $m_{PM} = \frac{3-1}{7-1} = \frac{2}{6} = \mathbf{\frac{1}{3}}$ ◄

3.4 $\tan P\hat{S}K = m_{PM} = \frac{1}{3}$ ➡ $P\hat{S}K = 18{,}43^\circ$

$\therefore\ \theta = \mathbf{71{,}57^\circ}$ ◄ . . . *∠s of ΔPSK*

3.5 In ΔPSK: $\cos\theta = \frac{PK}{PS}$

$\therefore \cos 71{,}57^\circ = \frac{3}{PS}$

$\therefore\ \mathbf{PS} = \frac{3}{\cos 71{,}57^\circ}$ . . . $a = \frac{k}{b}$ ➡ $b = \frac{k}{a}$

$\simeq$ **9,49 units** ◄

OR: $\sin 18{,}43^\circ = \frac{3}{PS}$, etc.

3.6 $N(x;\ y)$ on the line $y = -2x + 17$

➡ Point N is $(x;\ -2x + 17)$

$m_{NT} = m_{PM}$ . . . *NT || PM in trapezium*

$\therefore \frac{-2x+17-5}{x-(-1)} = \frac{1}{3}$

$\therefore \frac{-2x+12}{x+1} = \frac{1}{3}$

$\therefore -6x + 36 = x + 1$

$\therefore -7x = -35$

$\therefore x = 5$ & $y = -2(5) + 17 = 7$

$\therefore$ **N(5; 7)** ◄

OR: Find the equation of TN:

Substitute $m = \frac{1}{3}$ and $(-1;\ 5)$ in

$y - y_1 = m(x - x_1)$ OR $y = mx + c$

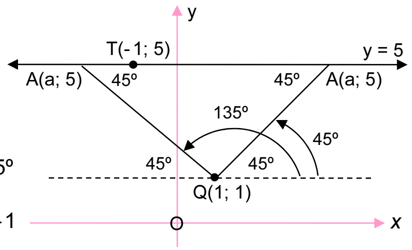
Equation is $y = \frac{1}{3}x + 5\frac{1}{3}$.

N is the point of intersection of TN and NP

$\therefore$ Solve the equations.

3.7.1 The equation:

y = 5 ◄



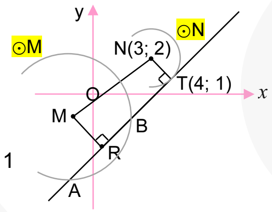
3.7.2 The gradient of AQ = tan 45º or tan 135º

$\therefore \frac{5-1}{a-1} = 1$ or -1

$\therefore \frac{4}{a-1} = \pm 1$

$\therefore a - 1 = \pm 4$ $\therefore$ **a = 5** or **–3** ◄

4.1 **M(–1; –1)** ◄



4.2 NT ⊥ AT . . . *tangent ⊥ radius*

$m_{NT} = \frac{1-2}{4-3} = -1$ ➡ $m_{AT} = 1$

Substitute m = 1 and T(4; 1) in:

$y - y_1 = m(x - x_1)$ or $y = mx + c$

$\therefore y - 1 = 1(x - 4)$ $\therefore 1 = (1)(4) + c$, etc.

$\therefore$ $\mathbf{y = x - 3}$ ◄

4.3 MR ⊥ AB . . . *MR is the line from the centre to the midpoint of chord AB*

$\therefore$ In ΔMRA: $AR^2 = MA^2 - MR^2$ . . . *Theorem of Pythagoras*

$= 9 - \left(\frac{\sqrt{10}}{2}\right)^2$ . . . $r^2 = 9$

$= 9 - \frac{10}{4}$

$= \frac{13}{2}$

$\therefore AR = \sqrt{\frac{13}{2}}$

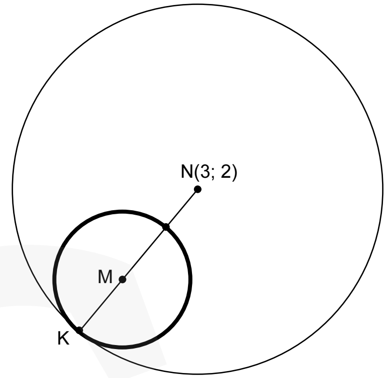
$\therefore AB = 2\sqrt{\frac{13}{2}}$ . . . $AB = 2AR$

$= \mathbf{\sqrt{26}\ units}$ ◄ . . . $\sqrt{4}\sqrt{\frac{13}{2}} = \sqrt{4 \times \frac{13}{2}}$

4.4 $MN^2 = (-1-3)^2 + (-1-2)^2 = 25$

$\therefore$ **MN = 5 units** ◄

4.5 This is 'another' circle ⊙N which touches ⊙M (internally).



MN = 5 units

& MK = 3 units

But NK, the radius of ⊙N = 5 + 3 = 8 units

$\therefore$ Equation of this ⊙N:

$(x-3)^2 + (y-2)^2 = 8^2$

$\therefore x^2 - 6x + 9 + y^2 - 4y + 4 = 64$

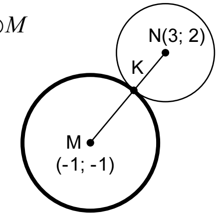
$\therefore$ $\mathbf{x^2 + y^2 - 6x - 4y - 51 = 0}$ ◄

There are 2 circles, centre N, which touch ⊙M.
The one mentioned earlier, where . . .

MN = 5 units . . . *in 4.4*

& MK = 3 units . . . *radius of ⊙M*

$\therefore$ KN, the radius of ⊙N = 2 units



$\therefore$ Equation of 'new' ⊙N:

$(x-3)^2 + (y-2)^2 = 2^2$

$\therefore x^2 - 6x + 9 + y^2 - 4y + \not{4} = \not{4}$

$\therefore$ $\mathbf{x^2 + y^2 - 6x - 4y + 9 = 0}$ ◄

This circle ⊙N touches circle ⊙M (externally).

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FORMULAE, GRADIENT & ANGLE OF INCLINATION

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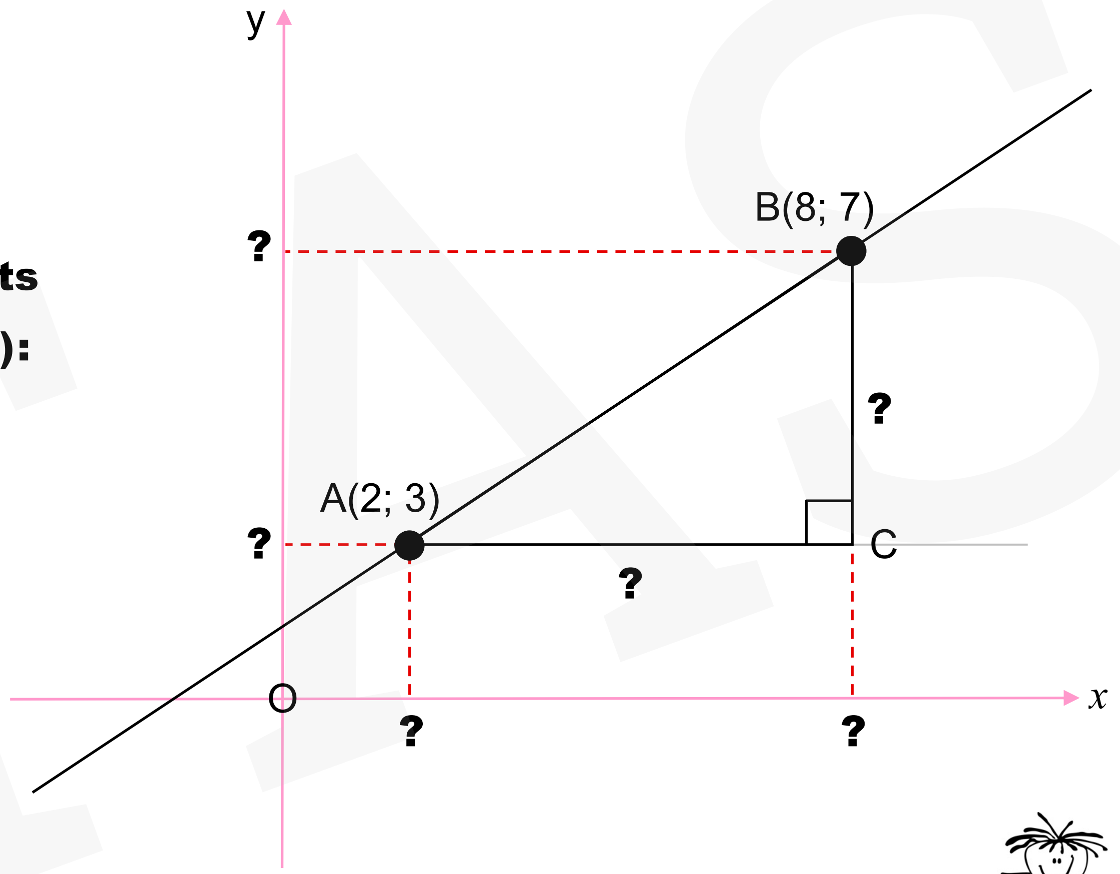
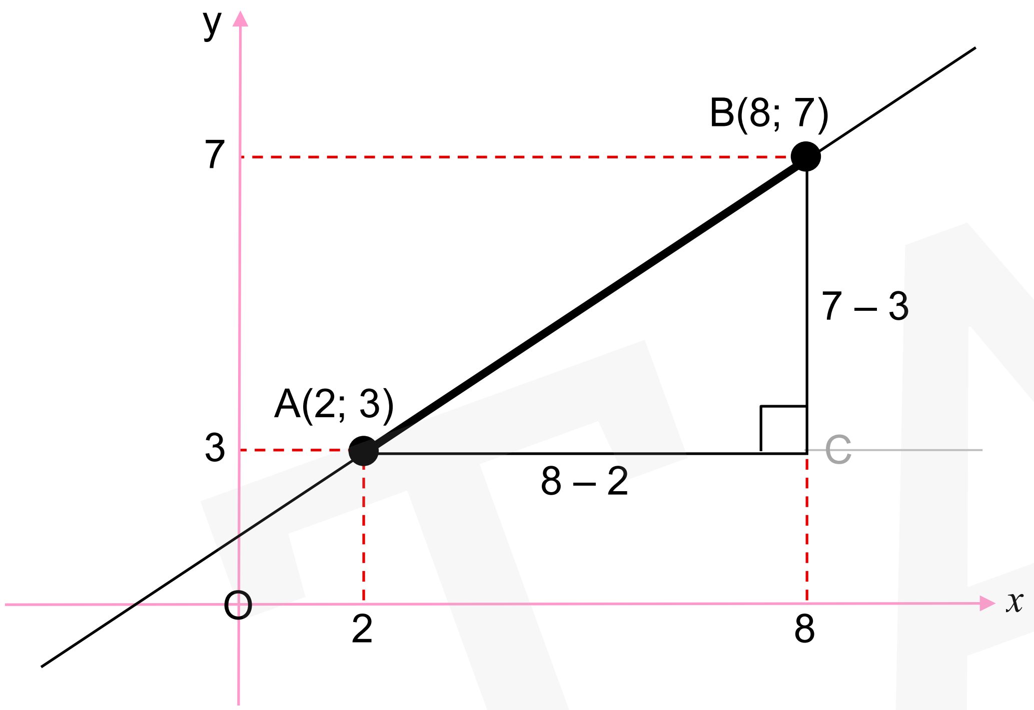
Formulae

See the two points

A(2; 3) & B(8; 7):



Length/Distance AB?



Note: The **horizontal length** AC and the **vertical length** BC

AB is the **hypotenuse**

The Theorem of Pythagoras:

$\therefore\ \mathbf{AB^2} = (8-2)^2 + (7-3)^2$

$= 36 + 16$

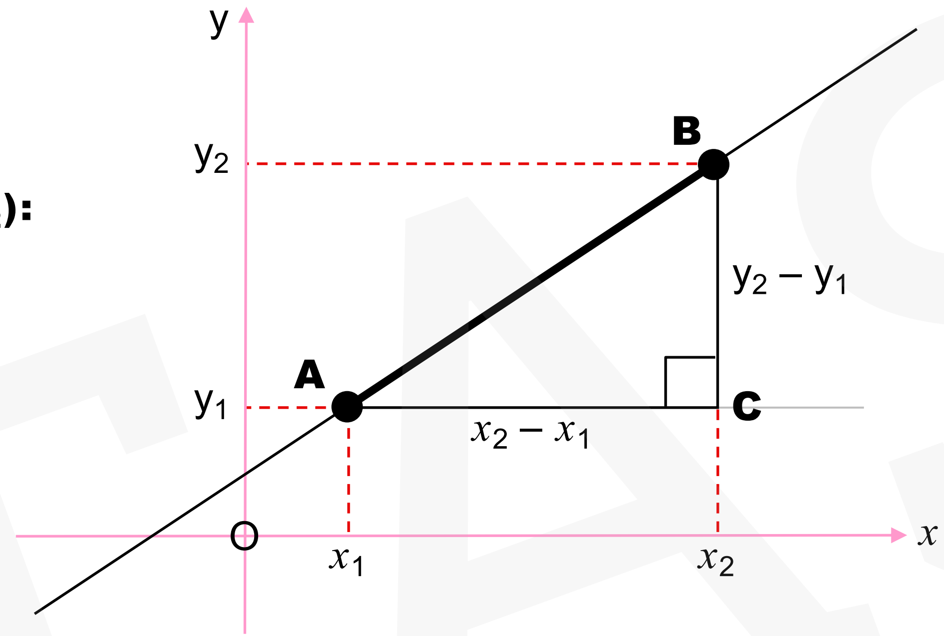
$= 52$

$\therefore\ \mathbf{AB} = \sqrt{52} = \sqrt{4 \times 13} = 2\sqrt{13}$ units

THE DISTANCE FORMULA

See the two points

A(x_1; y_1) and B(x_2; y_2):



AB is the **hypotenuse**

Horizontal length **AC**?

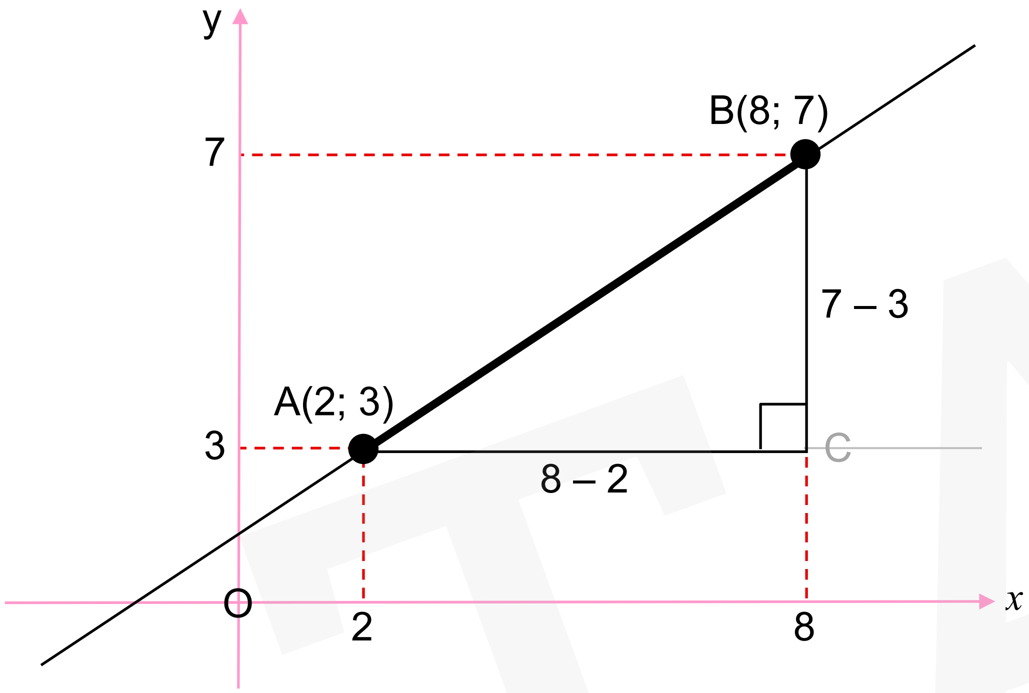
Vertical length **BC**?

$$AB^2 = AC^2 + BC^2 \quad \ldots \textit{ Theorem of Pythagoras}$$

$$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\therefore AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Gradient of line AB?



The gradient of line AB

$$= \frac{\text{the vertical length}}{\text{the horizontal length}}$$

$$= \frac{7 - 3}{8 - 2}$$

$$= \frac{4}{6}$$

$$= \frac{2}{3}$$

Note: The length of the **horizontal** line, AC

= the **difference between the x-coordinates**

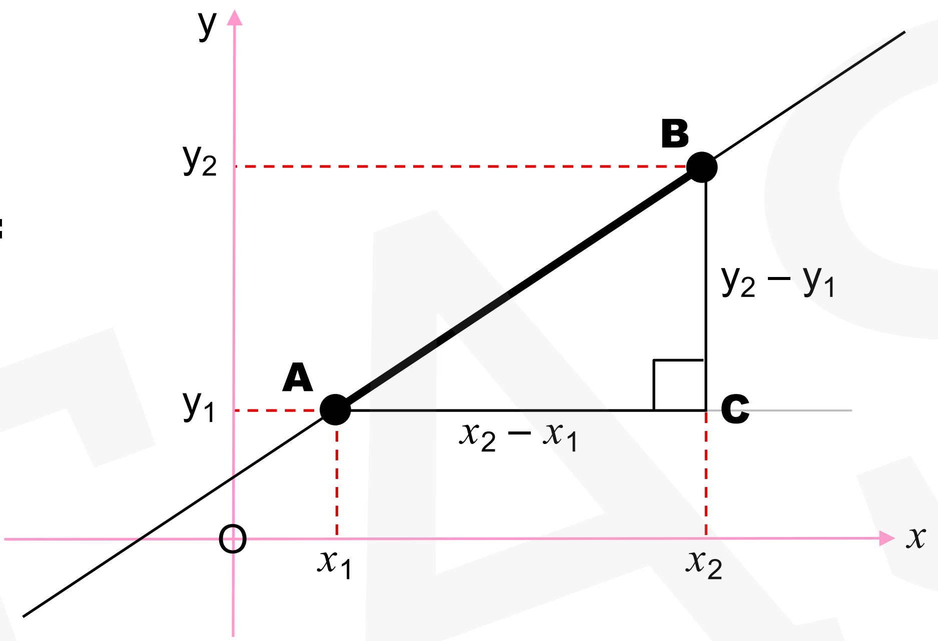
& The length of the **vertical** length, BC

= the **difference between the y-coordinates**

THE GRADIENT FORMULA

See the two points

A(x_1; y_1) and B(x_2; y_2):

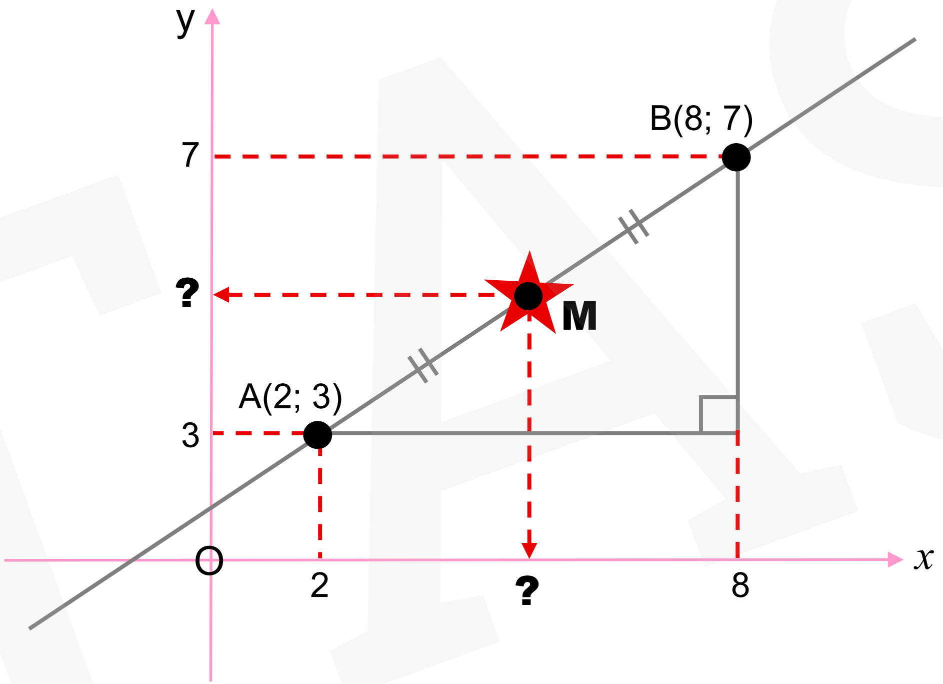


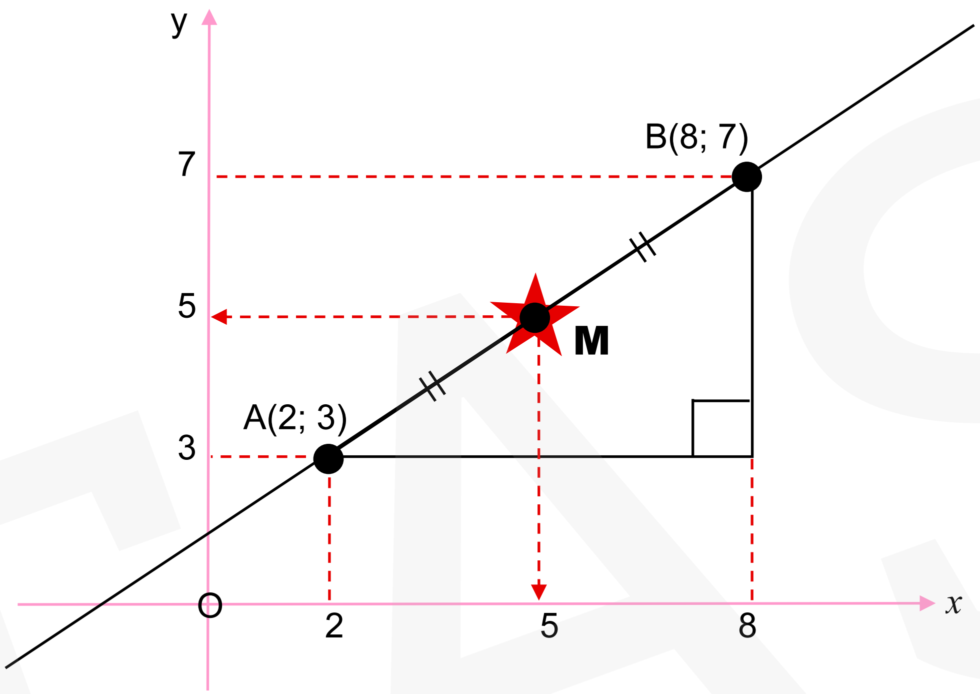
Horizontal length **AC**:
$x_2 - x_1$

Vertical length **BC**:
$y_2 - y_1$

The Gradient of line AB, m_{AB} $= \dfrac{BC}{AC} = \dfrac{\text{change in } y}{\text{change in } x} = \dfrac{\mathbf{y_2 - y_1}}{\mathbf{x_2 - x_1}}$

Midpoint of line segment AB

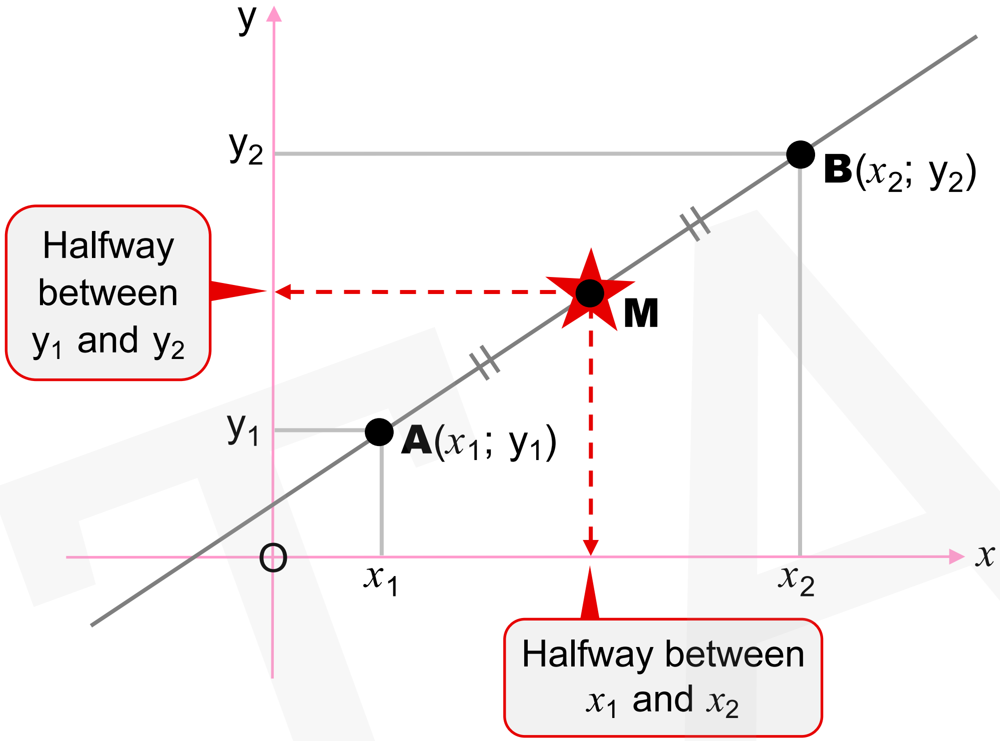




Note: $x_M = \dfrac{2 + 8}{2} = 5$ . . . the **average** of 2 and 8

& $y_M = \dfrac{3 + 7}{2} = 5$ . . . the **average** of 3 and 7

THE MIDPOINT FORMULA



The co-ordinates of the midpoint, **M**, *are the* **averages** *of the co-ordinates of the endpoints, A and B.*

The co-ordinates of the midpoint:

Note the PLUS signs

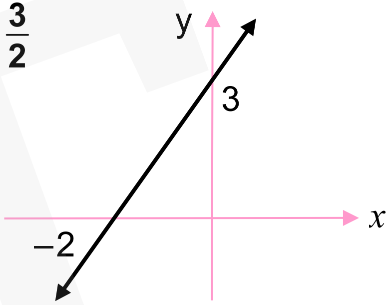
$$M\left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$$

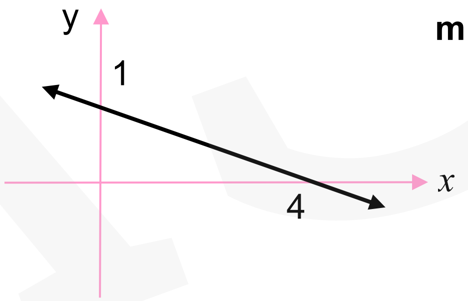
The Gradient of a line

Values

POSITIVE **NEGATIVE** **ZERO** **UNDEFINED**

$m = +\frac{3}{2}$

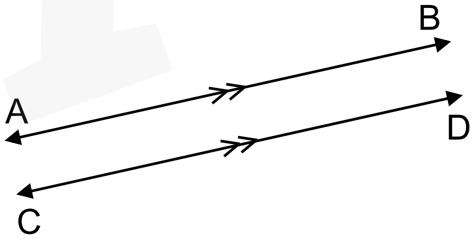




$m = -\frac{1}{4}$

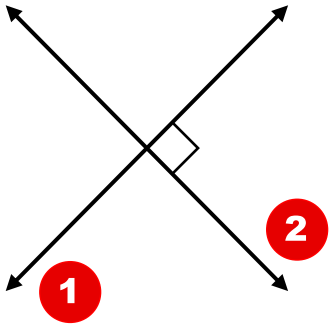
Parallel lines

Parallel lines have **equal** gradients.



AB || CD ➡ $m_{AB} = m_{CD}$

Perpendicular lines

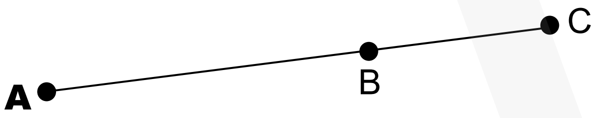


If the gradient of line ❶ is $\frac{2}{3}$, then the gradient of line ❷ will be $-\frac{3}{2}$

Note: $m_{1} \times m_{2} = \left(+\frac{2}{3}\right)\left(-\frac{3}{2}\right) = -1$

i.e. The **product** of the gradients of ⊥ lines is **−1**.

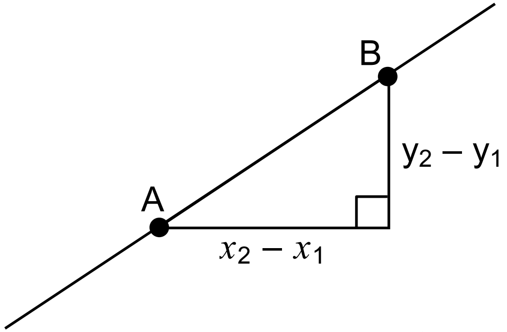
Collinear points



Three points A, B & C are collinear if the gradients of **A**B & **A**C are equal. (Note: Point **A** is common.)

$m_{AB} = m_{AC} \Longleftrightarrow$ A, B & C are collinear

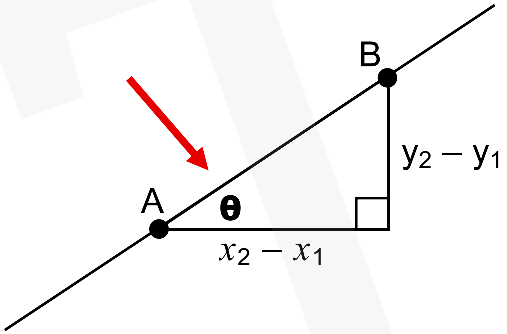
GRADIENT



$$\mathbf{m} = \frac{\text{change in y}}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

. . . *the* **gradient** *of the line*

Also:



$$\mathbf{tan}\ \boldsymbol{\theta} = \frac{\text{opposite}}{\text{adjacent}} = \frac{y_2 - y_1}{x_2 - x_1}$$

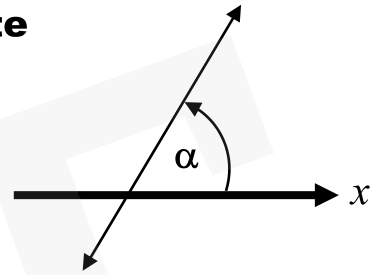
. . . *where θ is the* **angle of inclination** *of the line*

The Inclination of a line

Angles α and β below are **angles of inclination**.

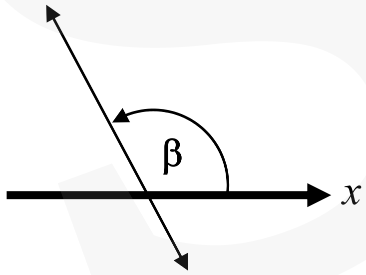
The Inclination of a line is the **angle** which the line makes with the positive direction of the x-axis.

α acute



gradient of the line is positive

β obtuse



gradient of the line is negative

Gradient, **m = tan** α or **tan** β
where α and β are the $\angle^s$ **of inclination**.

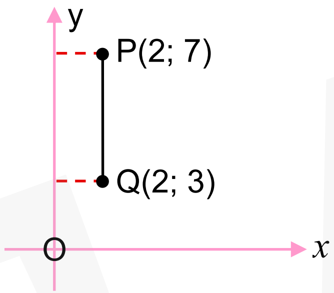
An Investigation

3 CASES of lines:

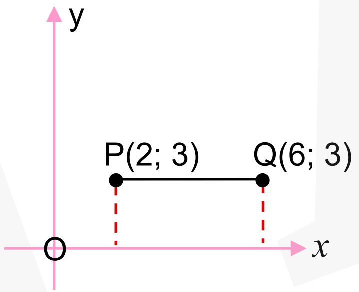
1. Vertical and Horizontal lines
2. Lines through (or from) the origin
3. Lines through any 2 given points

▶ Case 1: Vertical and Horizontal lines

1.1 P(2; 7) & Q(2; 3)



1.2 P(2; 3) & Q(6; 3)



For PQ, write down:

(a) the length

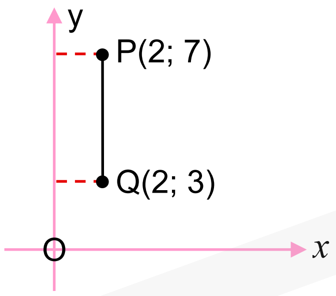
(b) the gradient

(c) the midpoint

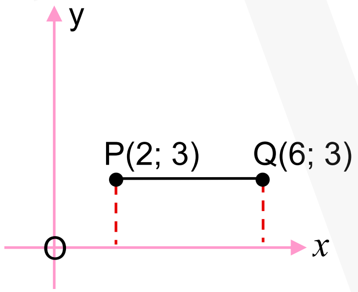
(d) the equation

ANSWERS

1.1 P(2; 7) & Q(2, 3)



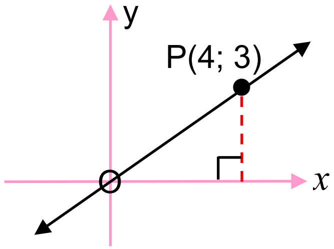
1.2 P(2; 3) & Q(6; 3)



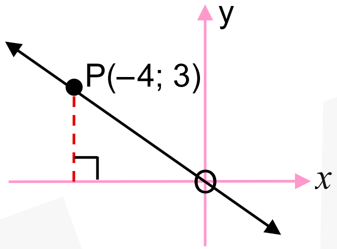
	1.1	**1.2**
(a) the length	4 units	4 units
(b) the gradient	undefined	0
(c) the midpoint	(2; 5)	(4; 3)
(d) the equation	$x = 2$	y = 3

▶ Case 2: Lines through (or from) the origin

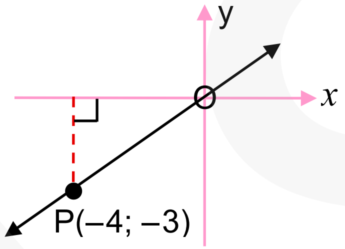
2.1



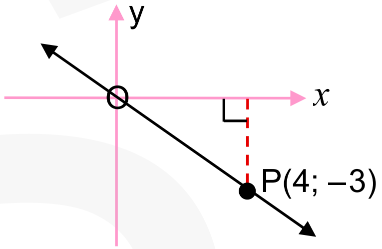
2.2



2.3



2.4



For OP, write down:

(a) the length

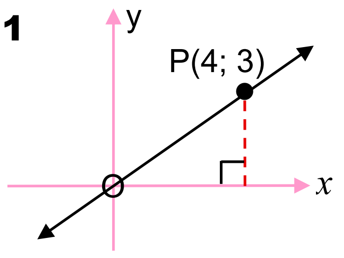
(b) the gradient

(c) the midpoint

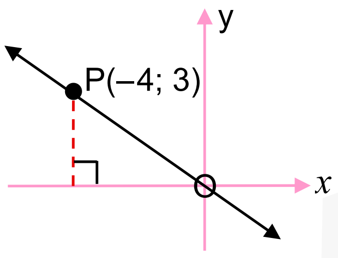
(d) the equation

ANSWERS

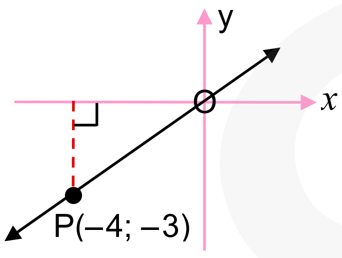
2.1



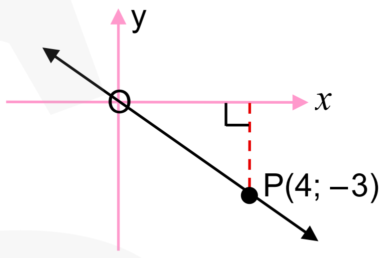
2.2



2.3



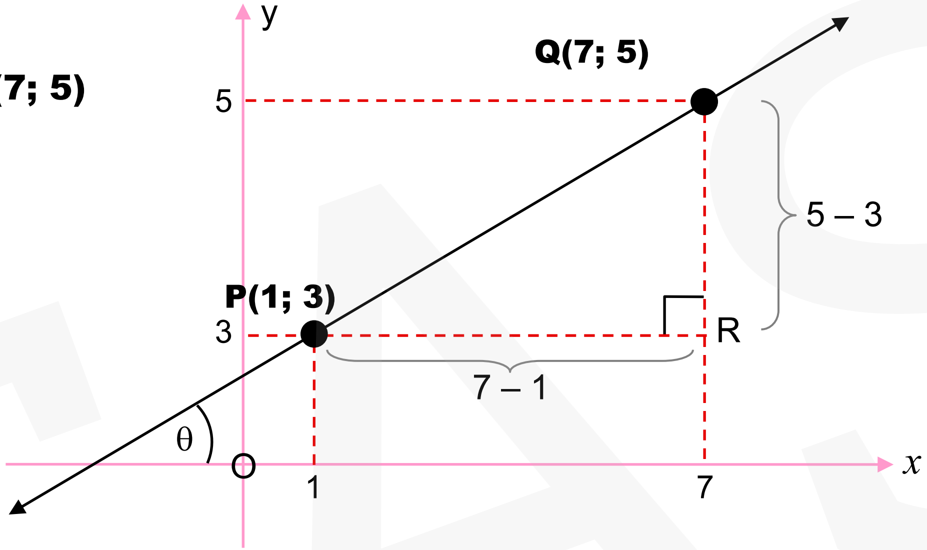
2.4



	2.1	**2.2**	**2.3**	**2.4**
(a) the length of OP:	5 units	5 units	5 units	5 units
(b) the gradient of OP:	$\frac{3}{4}$	$-\frac{3}{4}$	$\frac{3}{4}$	$-\frac{3}{4}$
(c) the midpoint of OP:	$\left(2;\ 1\frac{1}{2}\right)$	$\left(-2;\ 1\frac{1}{2}\right)$	$\left(-2;\ -1\frac{1}{2}\right)$	$\left(2;\ -1\frac{1}{2}\right)$
(d) the equation of OP:	$y = \frac{3}{4}x$	$y = -\frac{3}{4}x$	$y = \frac{3}{4}x$	$y = -\frac{3}{4}x$

▶ Case 3: Lines through any 2 points

See points **P(1; 3)** and **Q(7; 5)**



▶ Distance/Length, PQ

$PQ^2 = (7-1)^2 + (5-3)^2$ . . . *Theorem of Pythagoras in ΔPQR*

$= 6^2 + 2^2$

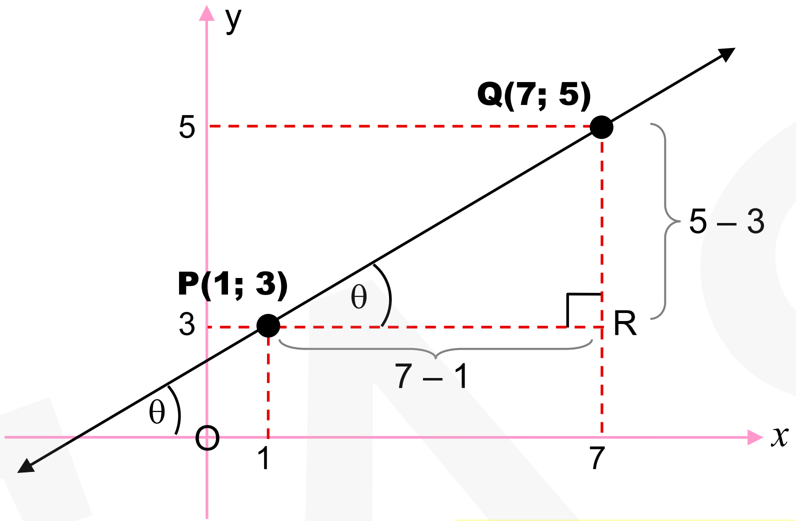
$= 40$

$\therefore PQ = \sqrt{40} = \sqrt{4 \times 10} = 2\sqrt{10}$ units

$$PQ^2 = (x_Q - x_P)^2 + (y_Q - y_P)^2$$

$$\therefore PQ = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2}$$

Gradient



The gradient of line PQ,

$m_{PQ} = \frac{1}{3}$

The gradient of line PQ $= \frac{y_Q - y_P}{x_Q - x_P}$

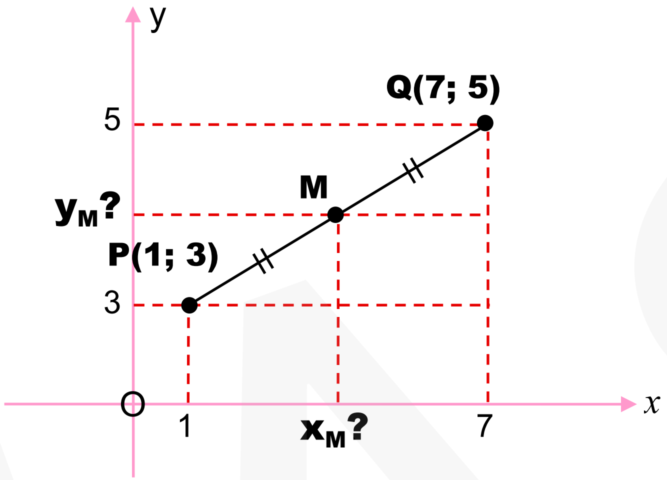
The ∠ of inclination ?

$\tan\theta = \frac{QR}{PR} = m_{PQ} = \frac{1}{3}$

∴ **The ∠ of inclination**, $\theta = 18{,}43^\circ$ . . . $\tan^{-1}\left(\frac{1}{3}\right)$

The gradient of the line is the tan of the ∠ of inclination
i.e. $m_{PQ} = \tan\theta$

▶ Midpoint



We can often **just write down** the coordinates of a midpoint **by inspection**!

The midpoint, M, of PQ:

- x_M is halfway between 1 and 7,

$\therefore\ x_M = \frac{1+7}{2} = 4$ . . . the **average** of 1 & 7

- y_M is halfway between 3 and 5,

$\therefore\ y_M = \frac{3+5}{2} = 4$ . . . the **average** of 3 & 5

$\therefore$ Midpoint, M: (4; 4)

The midpoint of PQ $= \left(\frac{x_P + x_Q}{2}\ ;\ \frac{y_P + y_Q}{2}\right)$

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STRAIGHT LINE GRAPHS

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GRAPH CONCEPTS

1 : Axis intercepts

On the **y-axis**: $\mathbf{x = 0}$.

On the **x-axis**: $\mathbf{y = 0}$.

2 : The Equations of the Axes

The **equation** of a graph is true for **all** points on the graph.

The **y-axis**: $\mathbf{x = 0}$; & The **x-axis**: $\mathbf{y = 0}$.

3 : Types of graph

e.g. $\mathbf{y = mx + c}$ a straight line ; $\mathbf{x^2 + y^2 = r^2}$ a circle ; $\mathbf{y = ax^2 + bx + c}$ a parabola

GRAPH CONCEPTS cont . . .

FACT 1 : Points on Graphs

FACT 2 : Point(s) of Intersection

Found:

"algebraically" by solving the 2 equations,

or **"graphically"** by reading from the graph.

And now, Straight Line Graphs . . .

Equations of Straight Line Graphs

- Standard form(s)
- General form
- Non-standard forms

Standard form

The standard form of the equation of a straight line is:

$$y = mx + c:$$

where **m** = the gradient & **c** = the y-intercept

- Lines || x-axis: **y = c** . . . (m = 0) . . . HORIZONTAL LINES
- Lines through the origin: **y = mx** . . . (c = 0)

BUT THEN . . .

- Lines || y-axis: **x = k** . . . VERTICAL LINES

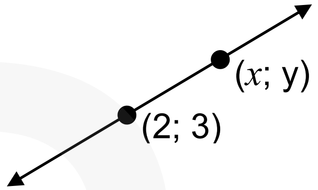
This is an equation which doesn't 'fit' any 'standard form'.

OR, use:

$$\mathbf{y - y_1 = m(x - x_1)}$$

An explanation

Given a fixed point, e.g. (2; 3), on a line, then, for **any** other point (x; y) on the line, it is **true** that:

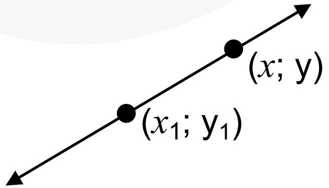


$$\frac{y - 3}{x - 2} = m \quad \ldots = \textit{the gradient of the line}$$

$$\therefore \; y - 3 = m(x - 2)$$

So, generally . . .

Given a **fixed** point (x_1; y_1), then, for **any** point (x; y) on the line, it is true that:



$$\frac{\mathbf{y - y_1}}{\mathbf{x - x_1}} = \mathbf{m} \quad \ldots = \textit{the gradient of the line}$$

$$\therefore \; \mathbf{y - y_1 = m(x - x_1)}$$

BE OPEN TO THIS ALTERNATIVE TO $y = mx + c$.

It is a much quicker method!

Finding the equation of a line . . . *(minimum requirements)*

Worked Example 1

1 Given m and c :

The gradient/intercept method

1.1 **Given:** A line has a gradient of –2 and cuts the y-axis at 3.

1.2 **Given:** A line || to the line y = $-x$ + 2, passes through the point (0; 4).

1.3 **Given:** A line through the origin, making an angle of 45º with the x-axis.

Worked Example 2

2 Given m and a point :

2.1 **Given:** A line has a gradient of 3 and passes through the point (1; 7).

2.2 **Given:** A line passes through point (–2; 4) and is perpendicular to line y = 2x + 5.

Worked Example 3

3 Given 2 points :

3.1 **Given:** A line passes through the points (–3; 1) and (4; –6).

3.2 **Given:** A line passes through points (–3; –2) and (–3; 5).

3.3 **Given:** A line passes through points (–2; –5) and (4; –5).

Worked Example 1

1 **Given m and c :**

1.3 **Given:** A line through the origin, making an angle of 45º with the x-axis.

Answer The 'standard form' of the equation is **y = mx** . . . **c = 0 !!!**
& the gradient, m = tan 45º = 1 . . . *gradient = the tan of the ∠ of inclination*

Equation: $y = x$ ➤

Worked Example 2

2 **Given m and a point :**

See **both** methods!

2.1 **Given:** A line has a gradient of 3 and passes through the point (1; 7).

Answer Substitute m = 3 & (1; 7) in:

$\mathbf{y = mx + c}$	OR	$\mathbf{y - y_1 = m(x - x_1)}$
$\therefore\ 7 = (3)(1) + c$		$\therefore\ y - 7 = 3(x - 1)$
$\therefore\ 4 = c$		$\therefore\ y - 7 = 3x - 3$
Equation: $y = 3x + 4$ ➤		$\therefore\ y = 3x + 4$ ➤

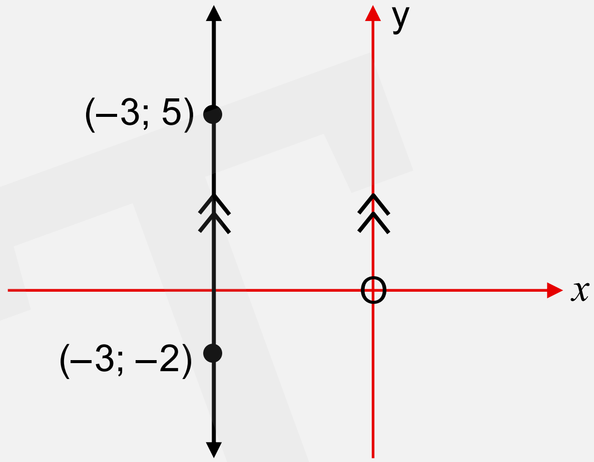
Worked Example 3

3 **Given** **2 points** **:**

3.2 **Given:** A line passes through points (–3; –2) and (–3; 5).

Answer

No 'method' needed! . . . *Draw a sketch!*



Remember to sketch the situation and think before being lead blindly by formulae and rote methods.

NB: The x-coordinates are the same!

∴ The line is parallel to the y-axis.

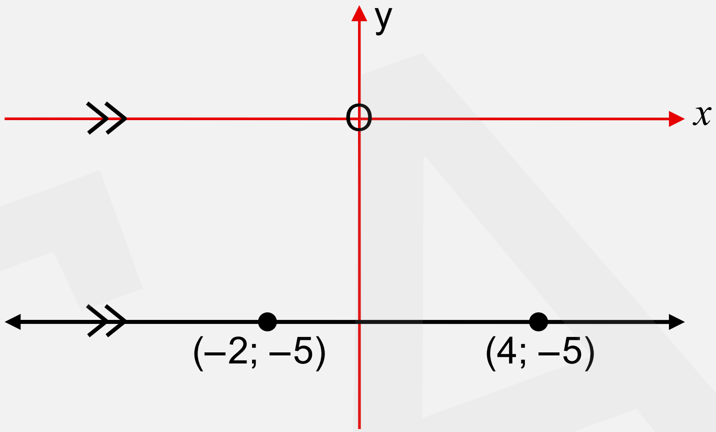
∴ Calculating m is 'not possible' . . . *The gradient is undefined!*

Equation: $x = -3$ ➤ . . . *'Standard form'* **x = k**

3.3 **Given:** A line passes through points (–2; –5) and (4; –5).

Answer

No 'method' needed! . . . *Draw a sketch!*



NB: The y-coordinates are the same!

∴ The line is parallel to the x-axis.

∴ The gradient is zero

Equation: $y = -5$ ◄ . . . *'Standard form'* **y = c**

The General form of the equation of a line

ax + by + c = 0 is the **General form** of the equation of a straight line.

This form is useful when finding the axis-intercepts and even the gradient.

Worked Example 4

(a) Draw the graph of the line with the equation

$$2x + 3y + 6 = 0$$

Show the axis-intercepts.

(b) What is the gradient of this line?

Worked Example 4

Draw the graph of the line with the equation

$$2x + 3y + 6 = 0$$

We use the 'dual-intercept' method.

Answer

(a) The axis-intercepts:

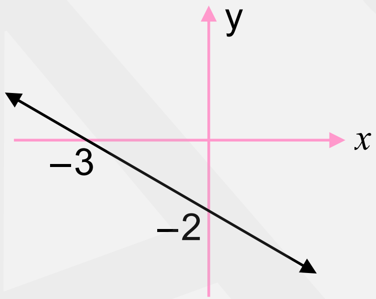
y-int: Put $\mathbf{x = 0}$, then $3y + 6 = 0$

$\therefore\ y = -2$

x-int: Put $\mathbf{y = 0}$, then $2x + 6 = 0$

$\therefore\ x = -3$

$\therefore$ The intercepts are: $(0; -2)$ and $(-3; 0)$



NB: There is NO need to convert this equation to standard form!

(b) The gradient $= -\dfrac{2}{3}$

The **General** form: $2x + 3y + 6 = 0$ converts to the **Standard** form: $y = -\dfrac{2}{3}x - 2$

And now . . . .

▸ Non-standard forms of the equation

Worked Example 5

(a) Draw the graphs of the lines with the following equations:

(1) $3x - 4y = 12$

(2) $\frac{x}{3} + \frac{y}{5} = 1$

(b) In each case, write down the gradient (m) and the y-intercept (c)

▸ The Dual-intercept method . . .

Worked Example 5

(a) Draw the graphs of the lines with the following equations:

(1) $3x - 4y = 12$ | (2) $\frac{x}{3} + \frac{y}{5} = 1$

(b) In each case, write down the gradient (m) and the y-intercept (c)

▶ **The Dual-intercept method . . .**

It is **not** necessary to convert these equations into the standard form, **y = mx + c**.

Answer

(a) To sketch these graphs, one can determine the intercepts as follows.

For the **y-intercept, put x = 0**

(1) $3(0) - 4y = 12$

$\therefore\ y = -3$

(2) $\frac{0}{3} + \frac{y}{5} = 1$

$\therefore\ y = 5$

& for the **x-intercept, put y = 0**

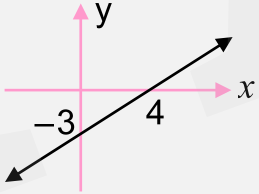
(1) $\therefore\ 3x - 4(0) = 12$

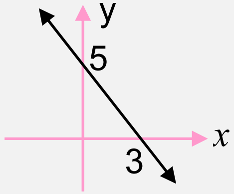
$\therefore\ x = 4$

(2) $\frac{x}{3} + \frac{0}{5} = 1$

$\therefore\ x = 3$

$\therefore$ **The sketches:**





(b) $m = \frac{3}{4}$ & $c = -3$ | $m = -\frac{5}{3}$ & $c = 5$

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TRIANGLES & QUADRILATERALS

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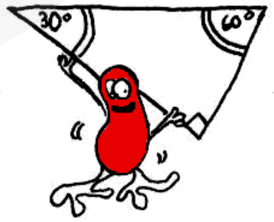
TRIANGLES

- Sum of Interior $\angle^s$
- Exterior $\angle$ of Δ

- Isosceles Δ^s
- Equilateral Δ^s
- Rt-$\angle^d$ Δ^s (Thm of Pythagoras)

- Area of a Δ and related facts



- Similar Δ^s
- Congruent Δ^s

- Midpoint Theorem

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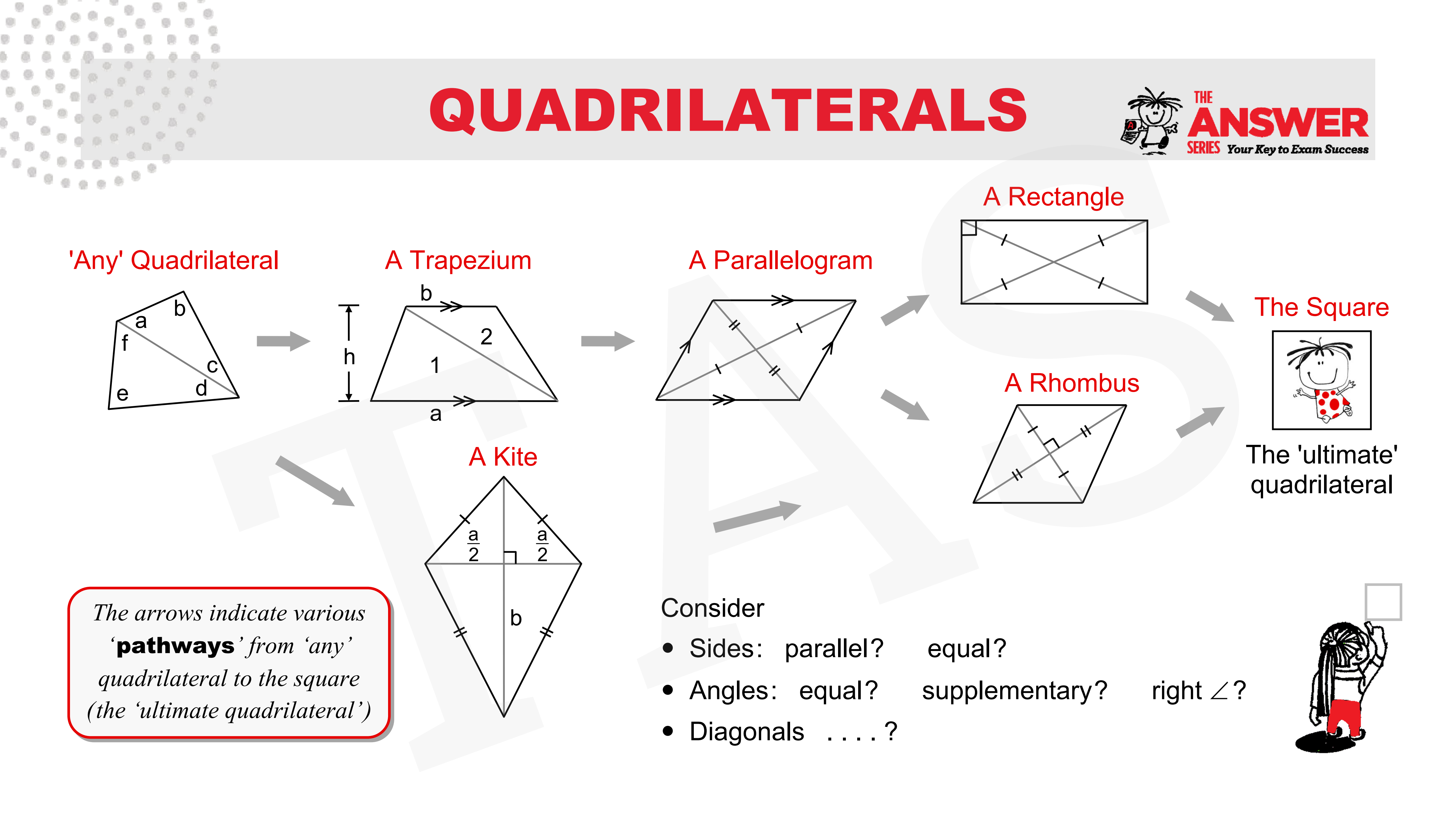
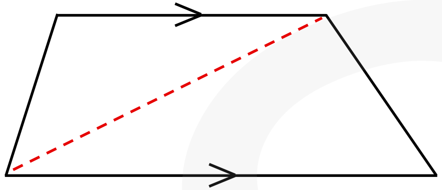
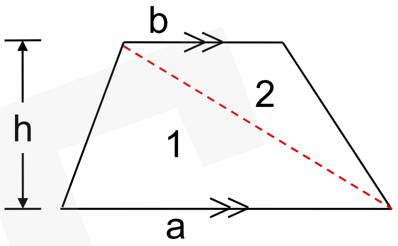


QUADRILATERALS
THE ANSWER SERIES Your Key to Exam Success
'Any' Quadrilateral
a
b
f
c
e
d
A Trapezium
b
h
2
1
a
A Parallelogram
A Rectangle
A Rhombus
The Square
The 'ultimate' quadrilateral
A Kite
a
2
a
2
b
The arrows indicate various 'pathways' from 'any' quadrilateral to the square (the 'ultimate quadrilateral')
Consider
• Sides: parallel? equal?
• Angles: equal? supplementary? right ∠?
• Diagonals . . . . ?

A Trapezium

- Can you derive a formula for the area of a trapezium?





The **Area** $= \Delta 1 + \Delta 2$

$= \frac{1}{2}ah + \frac{1}{2}bh$

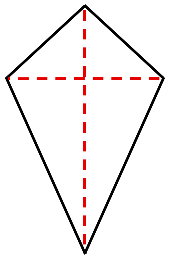
$= \mathbf{\frac{1}{2}(a + b).h}$

$\therefore$ The area of a trapezium:

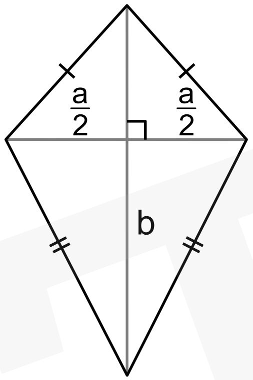
Half the sum of the || sides $\times$ the distance between them.

A Kite

Can you derive a formula for the area of a kite?



A Kite

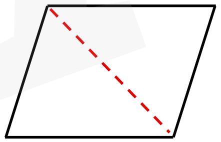


Given diagonals a and b . . .

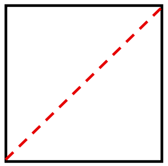
$$\textbf{Area} = 2\Delta^s = 2\left(\frac{1}{2}b \cdot \frac{a}{2}\right) = \frac{ab}{2} \quad \ldots \quad \frac{\text{the product of the diagonals}}{2}$$

$\therefore$ The area of a kite: **'Half the product of the diagonals'**

Could this formula apply to a **rhombus**?

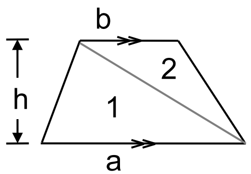


And to a **square**?



SUMMARY: AREAS

A Trapezium



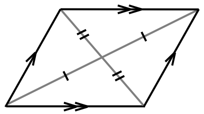
Area = $\Delta 1 + \Delta 2$

$= \frac{1}{2}ah + \frac{1}{2}bh$

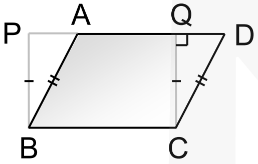
$= \mathbf{\frac{1}{2}(a + b).h}$

'Half the sum of the || sides × the distance between them.'

A Parallelogram



Area = base × height



$||^m$ ABCD = **ABCQ** + ΔQCD

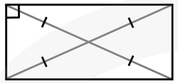
rect. PBCQ = **ABCQ** + ΔPBA

where ΔQCD ≡ ΔPBA . . . *RHS/90°HS*

∴ $||^m$ ABCD = rect. PBCQ (in area)

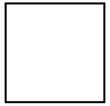
= BC × QC

A Rectangle



Area = ℓ × b

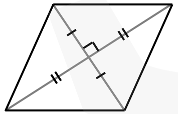
The Square



Area = s²

Since a square is a rectangle, a rhombus, a parallelogram, a kite, . . . ALL the properties of these quadrilaterals apply.

A Rhombus



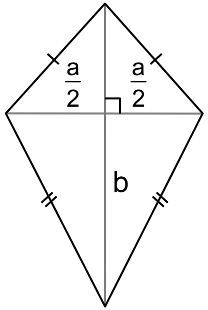
Area of a rhombus

= $\mathbf{\frac{1}{2}}$ **product of diagonals** (as for a kite)

or

= **base × height** (as for a parallelogram)

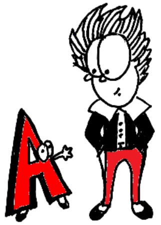
A Kite



Given diagonals a and b

Area = $2\Delta^s = 2\left(\frac{1}{2}b \cdot \frac{a}{2}\right) = \mathbf{\frac{ab}{2}}$

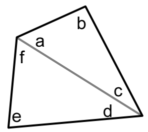
'Half the product of the diagonals'



QUADRILATERALS - definitions, areas & properties

All you need to know

'Any' Quadrilateral



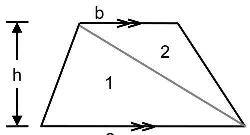
Sum of the $\angle^s$ of *any quadrilateral* = 360°

Sum of the interior angles
= (a + b + c) + (d + e + f)
= 2 × 180° . . . (2 Δ^s)
= 360°

*The arrows indicate various '**pathways**' from 'any' quadrilateral to the square (the 'ultimate quadrilateral'). These pathways, which combine logic and fact, are essential to use when proving specific types of quadrilaterals.*

See how the properties accumulate as we move from left to right, i.e. the first quad. has no special properties and each successive quadrilateral has all preceding properties.

A Trapezium

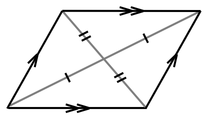


DEFINITION:
Quadrilateral with **1 pair of opposite sides ||**

Area = Δ 1 + Δ 2
= $\frac{1}{2}$ah + $\frac{1}{2}$bh
= $\mathbf{\frac{1}{2}}$ **(a + b).h**

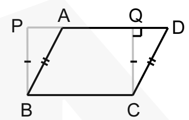
'Half the sum of the || sides × the distance between them.'

A Parallelogram



DEFINITION:
Quadrilateral with **2 pairs opposite sides ||**

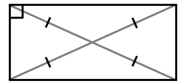
Area = base × height



$||^m$ ABCD = ABCQ + ΔQCD
rect. PBCQ = ABCQ + ΔPBA
where ΔQCD ≡ ΔPBA . . . *RHS/90°HS*
∴ $||^m$ ABCD = rect. PBCQ (in area)
= BC × QC

Properties:
2 pairs opposite sides equal
2 pairs opposite angles equal
*& **DIAGONALS BISECT ONE ANOTHER***

A Rectangle

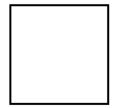


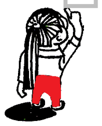
DEFINITION:
A $||^m$ with one right ∠

Area = ℓ × b

***DIAGONALS** are EQUAL*

The Square

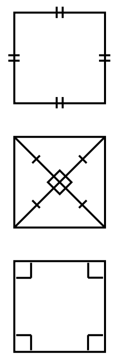




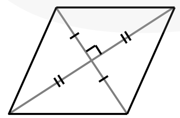
the 'ultimate' quadrilateral!

Area = s²
or
Area = $\mathbf{\frac{1}{2}}$ **product of diagonals**
(as for a kite)

Properties:
Since a square is a rectangle, a rhombus, a parallelogram, a kite, . . . ALL the properties of these quadrilaterals apply.



A Rhombus

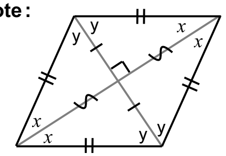


DEFINITION:
A $||^m$ with one pair of adjacent sides equal

Area
= $\mathbf{\frac{1}{2}}$ **product of diagonals** (as for a kite)
or
= **base × height** (as for a parallelogram)

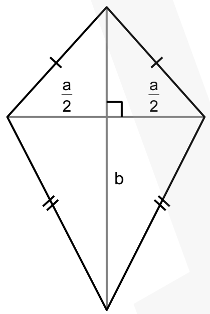
THE DIAGONALS
- *bisect one another PERPENDICULARLY*
- *bisect the angles of the rhombus*
- *bisect the area of the rhombus*

Note:



2x + 2y = 180° . . . $\angle^s$ *of* Δ *or co-int.* $\angle^s$; *|| lines*
➨ x + y = 90°

A Kite



DEFINITION:
Quadrilateral with **2 pairs of adjacent sides equal**

Given diagonals a and b
Area = $2\Delta^s = 2\left(\frac{1}{2}b.\frac{a}{2}\right) = \frac{ab}{2}$

'Half the product of the diagonals'

THE DIAGONALS
- *cut perpendicularly*
- ***ONE DIAGONAL** bisects the other diagonal, the opposite angles and the area of the kite*

Quadrilaterals play a prominent role in both Euclidean & Analytical Geometry Gr 10 – 12!

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TAS FET ANALYTICAL GEOMETRY COURSE

CIRCLES

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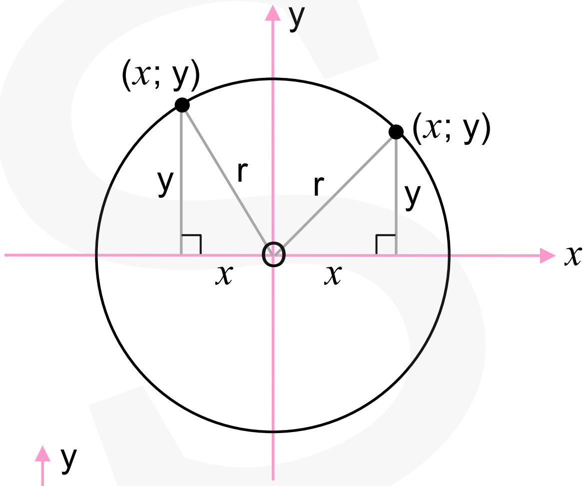
Equations of Circles

Circles with the origin as centre

True of **any** point **(x; y)** on a circle with centre **(0; 0)** and radius **r** is that:

$$x^2 + y^2 = r^2$$

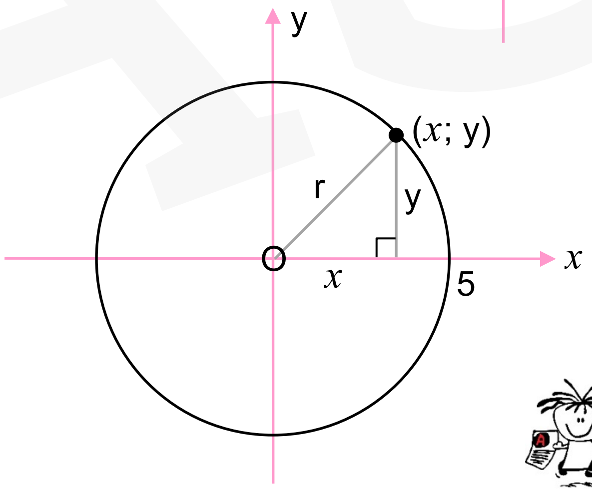
. . . *Theorem of Pythagoras!*



∴ This is the equation of the circle

Note: If the radius is 5 units, the equation of the circle is

$$x^2 + y^2 = 25$$



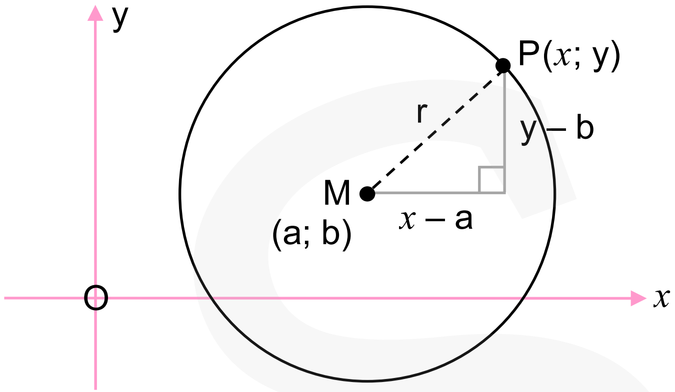
Circles with any given centre

True of any point **(x; y)** on a circle

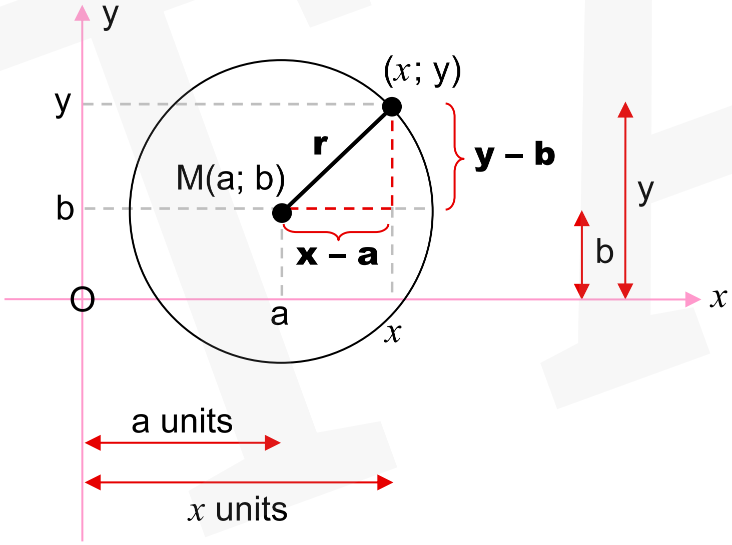
with centre **(a; b)** and radius **r** is that:

$$(x - a)^2 + (y - b)^2 = r^2$$

Distance formula! (Theorem of Pythagoras)



In detail



Applying the *Theorem of Pythagoras*:

The fact: $(x - a)^2 + (y - b)^2 = r^2$

is true for **any** point (x; y) on the circle.

& This is therefore the **standard form** of the equation for ALL CIRCLES, centre **(a; b)** and radius **r**.

Standard and General forms of the equations of lines and circles

Just as the equation of **A LINE** can be given as:

$y = -2x + 5$ *(standard form)* or $2x + y - 5 = 0$ *(general form)*

Standard form	⟷	**General form**
$y = mx + c$		$ax + by + c = 0$

so the equation of **A CIRCLE** can be given either way:

Standard form	⟷	**General form**
$(x - a)^2 + (y - b)^2 = r^2$		$Ax^2 + Bx + Cy^2 + Dy + E = 0$

$(x + 2)^2 + (y - 3)^2 = 16$ or $x^2 + 4x + y^2 - 6y - 3 = 0$

Conversions . . .

from:

◆ **Standard form** → **General form**

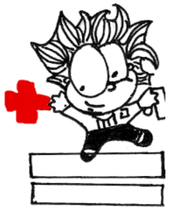
$$(x + 2)^2 + (y - 3)^2 = 16 \quad \text{. . . } \textit{standard form}$$

$$\therefore\ x^2 + 4x + 4 + y^2 - 6y + 9 = 16$$

$$\therefore\ x^2 + 4x + y^2 - 6y - 3 = 0 \quad \text{. . . } \textit{general form}$$

& Backwards:

◆ **General form** → **Standard form**



$$x^2 + 4x + y^2 - 6y - 3 = 0 \quad \text{. . . } \textit{general form}$$

$$\therefore\ x^2 + 4x + y^2 - 6y = 3$$

$$\therefore\ x^2 + 4x \mathbf{+ 4} + y^2 - 6y \mathbf{+ 9} = 3 \mathbf{+ 4 + 9} \quad \text{. . . COMPLETION OF SQUARES}$$

$$\therefore\ (x + 2)^2 + (y - 3)^2 = 16 \quad \text{. . . } \textit{standard form}$$

We need the **Standard form** to determine:
the centre, (–2; 3) and **the radius (4 units)** of the circle.

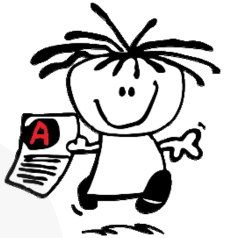
Completing the Square

Let's start at the beginning …

Squaring:

$$(x + 3)^2 = (x + 3)(x + 3) = x^2 + 3x + 3x + 9 = x^2 + 6x + 9$$

In reverse: $x^2 + 6x + 9 = (x + 3)^2$



Now, complete:

$(x - 4)^2 =$ ______________________________

& In reverse: ______________________

$(x + 5)^2 =$ ______________________________

& In reverse: ______________________

Try the following:

(a) $x^2 + 2x + (?)^2$:

$= (x \ldots\ldots.)^2$

(b) $x^2 - 8x + (?)^2$:

$= (x \ldots\ldots.)^2$

(c) $x^2 + 10x + (?)^2$:

$= (x \ldots\ldots.)^2$

(d) $x^2 - 14x + (?)^2$:

$= (x \ldots\ldots.)^2$

(e) $x^2 + 3x + (?)^2$:

$= (x \ldots\ldots.)^2$

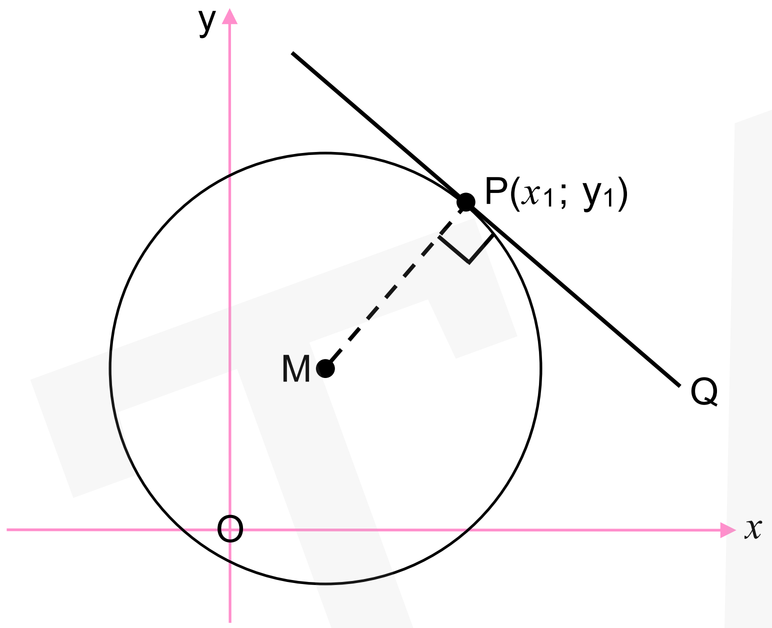
(f) $x^2 - x + (?)^2$:

$= (x \ldots\ldots.)^2$

To complete a square: Add $\left(\frac{1}{2}\textbf{ coefficient of x}\right)^2$

A Tangent to a circle . . .

is **perpendicular** to the **radius** of the circle at the **point** of **contact**.



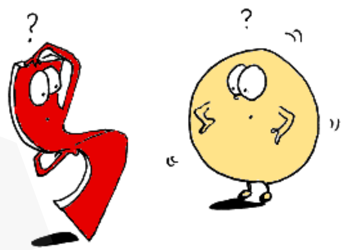
$m_{MP} = 2 \Rightarrow m_{PQ} = -\frac{1}{2}$

($\because$ radius MP ⊥ tangent PQ)

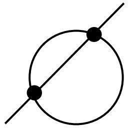
To find the **equation** of a tangent, use **"m and 1 point"** in the straight line equation:

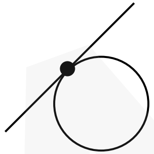
$$\mathbf{y - y_1 = m(x - x_1)}$$

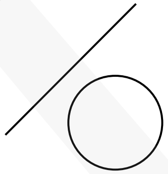
Point(s) of intersection of a Line and a Circle



A line and a circle **either**

① **"cut"** (twice!) [secant]  *(2 points in common)*

or ② **"touch"** (once!) [tangent]  *(1 point in common)*

or ③ **don't cut or touch**  *(no points in common)*

If we substitute $y = mx + c$ into the equation of the ⊙,

there will **either** be: ① **2 solutions**

or ② **1 solution**

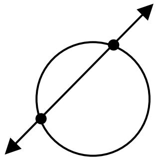
or ③ **no solutions**

for x, resulting in one of the above scenarios.

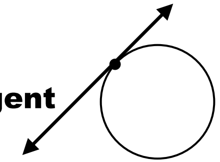
A link to Calculus . . .

It is important to integrate different topics in maths.

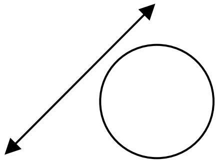
- Circle and a line

❶


2 points of intersection

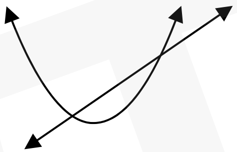
❷ **a tangent**


1 point of intersection

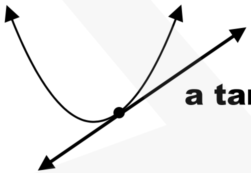
❸


no points of intersection

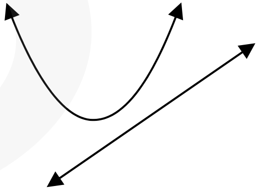
- Parabola and a line (Calculus)

❶


2 points of intersection

❷

a tangent

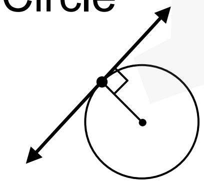
1 point of intersection

❸


no points of intersection

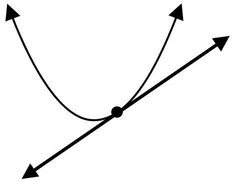
Finding the **equation of a tangent . . .**

to a Circle



&

to a Parabola (Calculus)

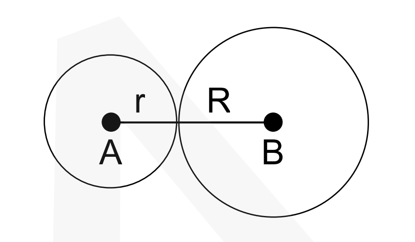


Two Circles . . .

An interesting fact . . .

When 2 ⊙'s touch,
the **distance between their centres**
= the SUM of their radii (& vice versa)

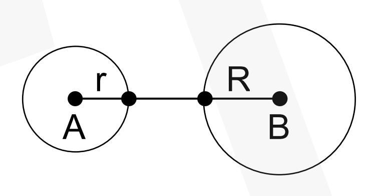
i.e. **AB = r + R**



. . . 1 point in common

We say these two circles touch externally.

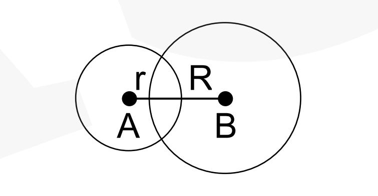
And . . .



AB > r + R

The circles do not cut or touch.

. . . No points in common



AB < r + R

The circles cut (twice).

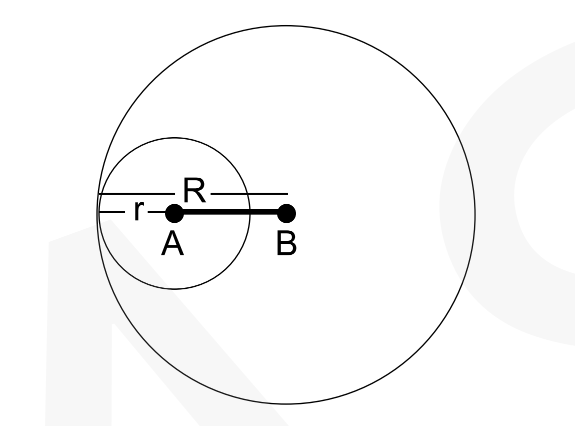
. . . 2 points in common

But, also consider the following . . .

When 2 ⊙'s touch **internally**,

the **distance between their centres**

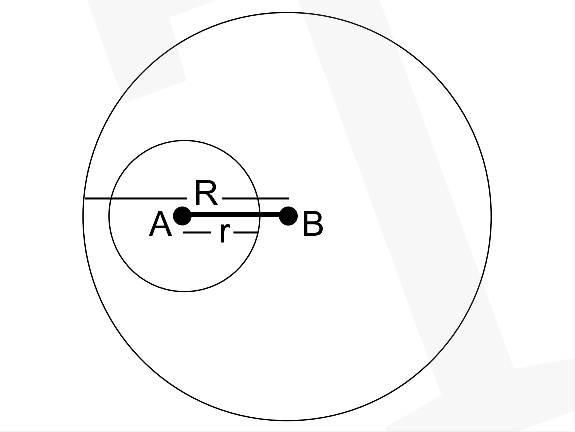
= the **DIFFERENCE** of their radii



AB = R – r

One point in common

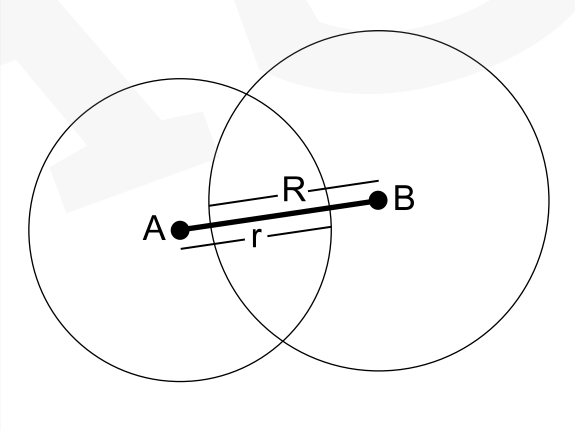
The circles do not touch



AB < R – r

No points in common

The circles cut twice



AB > R – r

2 points in common

Worked Example 1

The point $(x; 2)$ lies on the circle with centre the origin and radius $\sqrt{13}$.

(a) Write down the equation of the circle.

(b) Find the value(s) of x.

(c) Now sketch the circle and fill in all the intercept values.

Worked Example 1

The point (x; 2) lies on the circle with centre the origin and radius $\sqrt{13}$.

(a) Write down the equation of the circle.

(b) Find the value(s) of x.

(c) Now sketch the circle and fill in all the intercept values.

Answers

(a) $r^2 = \left(\sqrt{13}\right)^2 = 13$

Eqn. of ⊙ centre the origin:

$$x^2 + y^2 = r^2$$

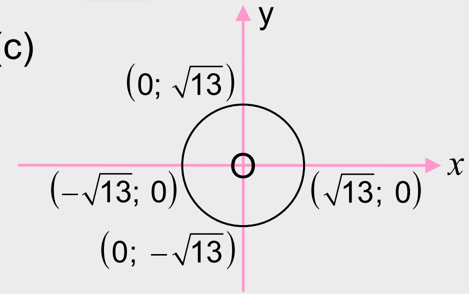
$\therefore$ Eqn. is: $x^2 + y^2 = 13$ ➤

(b) $x^2 + 2^2 = 13$

$\therefore\ x^2 = 9$

$\therefore\ x = \pm 3$ ➤

(c)



Worked Example 2

(a) Find the points of intersection of the **line** $y = x + 5$ & the **circle** $x^2 + y^2 = 25$ **graphically**.

(b) Confirm your answers **algebraically**.

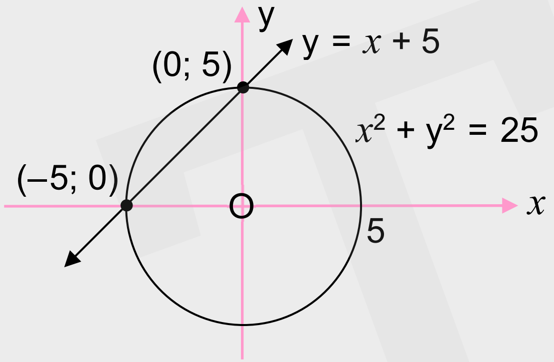
Worked Example 2

(a) Find the points of intersection of the **line** y = x + 5 & the **circle** $x^2 + y^2 = 25$ **graphically**.

(b) Confirm your answers **algebraically**.

Answers

(a) Graphically . . .



∴ The points of intersection (0; 5) & (–5; 0) ◄

(b) Algebraically . . .

To solve the equations, substitute y = x + 5 in $x^2 + y^2 = 25$:

$\therefore\ x^2 + (x + 5)^2 = 25$

$\therefore\ x^2 + x^2 + 10x + 25 = 25$

$\therefore\ 2x^2 + 10x = 0$

$\therefore\ 2x(x + 5) = 0$

$\therefore\ x = 0$ or -5

∴ y = 5 or 0 respectively . . . $y = x + 5$

∴ The points of intersection (0; 5) & (–5; 0) ◄

Often, when asked to find the **coordinates of a point (or points)**, it means finding a **point(s) of intersection** of two graphs, i.e. solving their equations simultaneously.

Worked Example 3

(a) Do the two graphs $2y - x + 4 = 0$ and $(x - 1)^2 + (y - 1)^2 = 10$ share any of their axis-intercepts?

(b) Now sketch these graphs on the same system of axes showing the centre of the circle and the points of intersection of the line and the circle.

Worked Example 3

(a) Do the two graphs $2y - x + 4 = 0$ and $(x - 1)^2 + (y - 1)^2 = 10$ share any of their axis-intercepts?

(b) Now sketch these graphs on the same system of axes showing the centre of the circle and the points of intersection of the line and the circle.

Answers

(a)

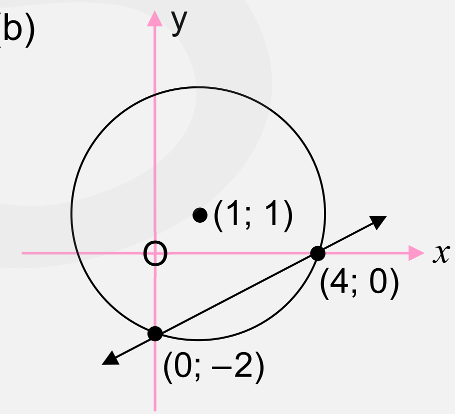
	$2y - x + 4 = 0$	$(x - 1)^2 + (y - 1)^2 = 10$
y-int.: *(Put $x = 0$)*	$\therefore 2y + 4 = 0$ $\therefore 2y = -4$ $\therefore y = \mathbf{-2}$	$\therefore 1 + (y - 1)^2 = 10$ $\therefore (y - 1)^2 = 9$ $\therefore y - 1 = \pm 3$ $\therefore y = 1 \pm 3$ $= 4$ or $\mathbf{-2}$

$\therefore$ Yes, they share a y-intercept: $(0; -2)$ ➤

x-int.: *(Put $y = 0$)*	$-x + 4 = 0$ $\therefore -x = -4$ $\therefore x = 4$	$(x - 1)^2 + 1 = 10$ Similarly: $x = 4$ or -2

$\therefore$ Yes, they share an x-intercept: $(4; 0)$ ➤

(b)



Worked Example 4

(a) Draw a sketch to determine whether the lines $y = 4$, $y = 5$ and $y = 6$ are tangents to the circle $x^2 + y^2 = 25$. Write down, where possible, any points of intersection.

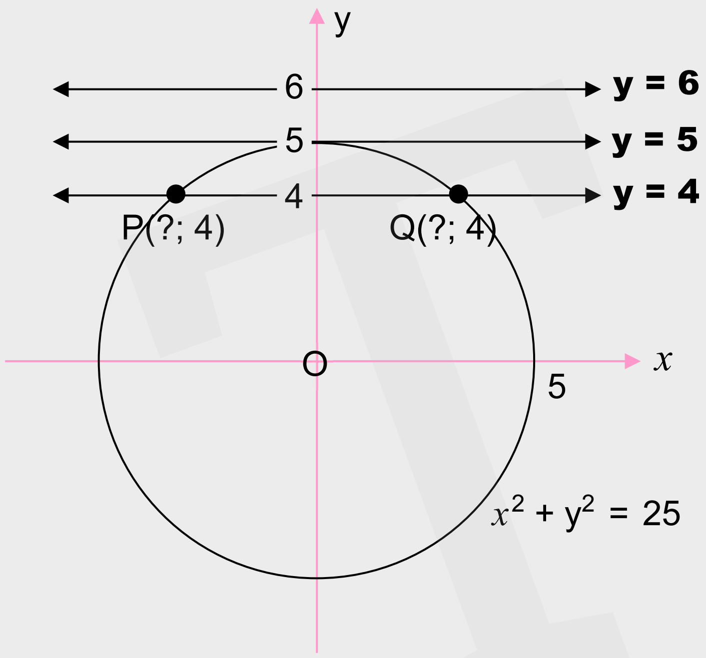
(b) Show how you would answer this question without a sketch, i.e. algebraically.

Worked Example 4

(a) Draw a sketch to determine whether the lines y = 4, y = 5 and y = 6 are tangents to the circle $x^2 + y^2 = 25$. Write down, where possible, any points of intersection.

Answers

(a)



- **y = 5 is a tangent** & the point of contact is (0; 5).

y = 6 and **y = 4** are not tangents

- **y = 6** does not *cut* or *touch* the circle.
- **y = 4** *cuts* the circle twice – at points P(–3; 4) & Q(3; 4)

By inspection, these coordinates satisfy the equation of the circle.

Worked Example 4

(b) Show how you would answer this question without a sketch, i.e. algebraically.

Answers

(b) We would find the point(s) of intersection (if any) by solving equations simultaneously:

$\mathbf{x^2 + y^2 = 25}$ and . . .

- **y = 4** ➡ $x^2 + 16 = 25$

  $\therefore\ x^2 = 9$

  $\therefore\ x = \pm 3$

  ∴ 2 points of intersection

  ∴ **y = 4** *cuts* the circle *twice*.

  ∴ pts. of int.: (-3; 4) & (3; 4)

- **y = 5** ➡ $x^2 + 25 = 25$

  $\therefore\ x^2 = 0$

  $\therefore\ x = 0$

  ∴ Only 1 point of intersection

  ∴ **y = 5** *touches* the circle *(once)*.

  ∴ point of contact: (0; 5)

- **y = 6** ➡ $x^2 + 36 = 25$

  $\therefore\ x^2 = -11$

  which is impossible
  (square always ≥ 0!)

  ∴ No point(s) of intersection

  ∴ **y = 6** doesn't *cut* or *touch* the circle.

Worked Example 5

(a) Is the line $y = -4x - 17$ a tangent to the circle with equation $x^2 + y^2 = 17$?

(b) If so, find the point of contact.

Worked Example 5

(a) Is the line $y = -4x - 17$ a tangent to the circle with equation $x^2 + y^2 = 17$?

(b) If so, find the point of contact.

Answers

(a) If the line and the circle do touch or cut each other, then, at the possible point(s) of intersection:

$y = -4x - 17$ **and** $x^2 + y^2 = 17$ . . . *the equations will be true simultaneously*

$\therefore\ x^2 + (-4x - 17)^2 = 17$

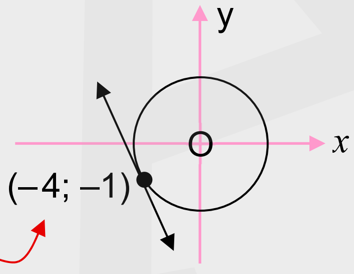
$\therefore\ x^2 + 16x^2 + 136x + 289 = 17$

$\therefore\ 17x^2 + 136x + 272 = 0$

$(\div 17)\quad \therefore\ x^2 + 8x + 16 = 0$

$\therefore\ (x + 4)^2 = 0$

$\therefore\ x = -4$



There is only 1 root ($\therefore$ Case 2)

$\therefore$ Yes, the line is a tangent to the circle

(b) Substitute $x = -4$:

$y = -4x - 17$

$\therefore\ y = -4(-4) - 17$

$= -1$

$\therefore$ The point of contact is $(-4; -1)$

Worked Example 6

The equation of a circle with radius $2\sqrt{2}$ units is:

$$x^2 + ax + y^2 - 8y + 12 = 0.$$

1. Determine the value(s) of a.
2. Write down the possible coordinates of the centre of the circle.

Worked Example 6

The equation of a circle with radius $2\sqrt{2}$ units is: $x^2 + ax + y^2 - 8y + 12 = 0$.

1. Determine the value(s) of a.
2. Write down the possible coordinates of the centre of the circle.

Answers

1.
$$x^2 + ax + y^2 - 8y = -12$$
$$\therefore\ x^2 + ax + \mathbf{\left(\frac{a}{2}\right)^2} + y^2 - 8y \mathbf{+ 16} = -12 \mathbf{+ \frac{a^2}{4} + 16}$$
$$\therefore\ \left(x + \frac{a}{2}\right)^2 + (y-4)^2 = \mathbf{4 + \frac{a^2}{4}}$$
$$\therefore\ 4 + \frac{a^2}{4} = \left(2\sqrt{2}\right)^2 \quad \ldots\ (= r^2)$$
$$\therefore\ \frac{a^2}{4} = 8 - 4$$
$$\therefore\ a^2 = 16$$
$$\therefore\ a = \pm 4$$ ◄

2. ∴ Centre: (–2; 4) or (2; 4) ◄ $\ldots \left(-\frac{a}{2}; 4\right)$ *where* $a = \pm 4$

Refer to the Answer Series Gr 12 Maths 2 in 1 pages xiii & xiv

ANALYTICAL GEOMETRY TOOLKIT

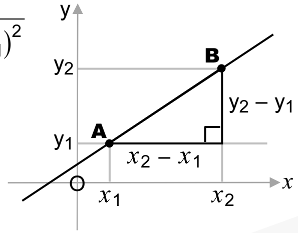
FORMULAE

Consider two points $A(x_1; y_1)$ and $B(x_2; y_2)$:

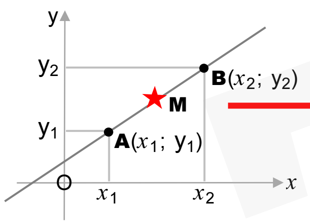
DISTANCE

$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ . . . *Thm of Pythagoras*

$\therefore AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



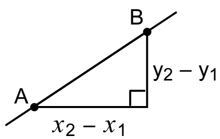
MIDPOINT



$M\left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$

The co-ordinates of the midpoint, M, are the ***averages*** *of the co-ordinates of the endpoints, A and B.*

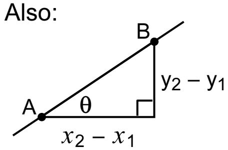
GRADIENT



$m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$

. . . *the* ***gradient*** *of the line*

Also:

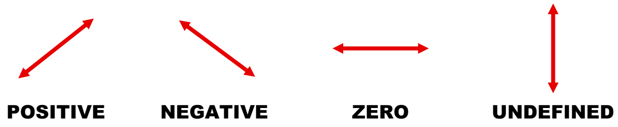


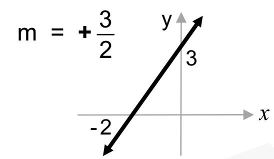
$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y_2 - y_1}{x_2 - x_1}$

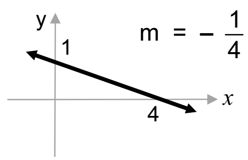
. . . *where θ is the* ***angle of inclination*** *of the line*

The Gradient of a line ↔ The Inclination of a line

Values

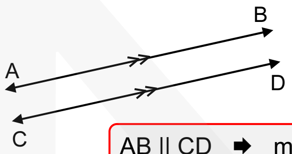






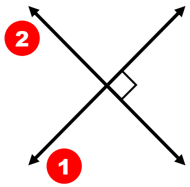
Parallel lines

Parallel lines have **equal** gradients.



AB || CD ➡ $m_{AB} = m_{CD}$

Perpendicular lines

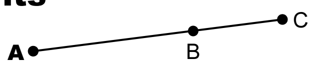


If the gradient of line ❶ is $\frac{2}{3}$, then the gradient of line ❷ will be $-\frac{3}{2}$

Note: $m_{❶} \times m_{❷} = \left(+\frac{2}{3}\right)\left(-\frac{3}{2}\right) = -1$

i.e. The **product** of the gradients of ⊥ lines is **−1**.

Collinear points



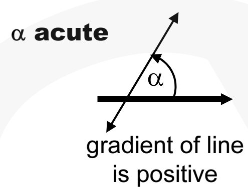
Three points A, B & C are collinear if the gradients of AB & AC are equal. (Note: Point **A** is common.)

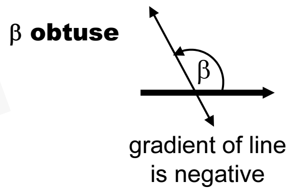
$m_{AB} = m_{AC}$ ⟺ A, B & C are collinear

The Inclination of a line

Angles α and β below are **angles of inclination**.

The inclination of a line is the **angle** which the line makes with the positive direction of the x-axis.





Gradient, **m = tan** α or **tan** β
where α and β are the ∠**s of inclination**.

GRAPH CONCEPTS

❶ : **Axis intercepts**

Every point on the **y-axis** has **x = 0**.
Every point on the **x-axis** has **y = 0**.

❷ : **The equation**

The **equation** of a graph is true for **all** points on the graph.

∴ The **equation** of the **y-axis** is **x = 0**;
& the **equation** of the **x-axis** is **y = 0**.

❸ : **Types of graph**

Different **types/patterns** are indicated by various equations.

e.g. $\mathbf{y = mx + c}$ indicates a straight line

$\mathbf{x^2 + y^2 = r^2}$ indicates a circle

$y = ax^2 + bx + c$ indicating a parabola

GRAPH CONCEPTS cont . . .

FACT 1 : Points on Graphs

If a point lies on a graph, the equation is true for its coordinates, i.e. the coordinates of the point satisfy the equation . . . so, substitute!

and, conversely,

If a point (i.e. its coordinates) satisfies the equation of a graph (i.e. "makes it true"), then it lies on the graph.

FACT 2 : Point(s) of Intersection

The coordinates of the point(s) of intersection of two graphs "obey the conditions" of both graphs, i.e. they satisfy both equations simultaneously.

They are found:

- **"algebraically"** by solving the 2 equations, or
- **"graphically"** by reading from the graph.

THESE 2 FACTS ARE CRUCIAL!

STRAIGHT LINE GRAPHS & their equations

Standard forms

- **$y = mx + c$:**

  where **m** = the gradient & **c** = the y-intercept

  When m = 0: **$y = c$** ... *a line* || **x-axis** ... **HORIZONTAL LINES**

  When c = 0: **$y = mx$** ... *a line through the* **origin**

  Also: **$x = k$** ... *a line* || **y-axis** ... **VERTICAL LINES**

- **$y - y_1 = m(x - x_1)$:**

  where **m** = the gradient & **$(x_1; y_1)$** is a fixed point.

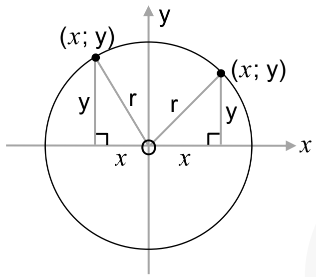
General form

The **general form** of the equation of a straight line is **$ax + by + c = 0$**, e.g. $2x + 3y + 6 = 0$

CIRCLES & their equations

Circles with the origin as centre

True of **any** point **(x; y)** on a circle with centre **(0; 0)** and radius **r** is that:

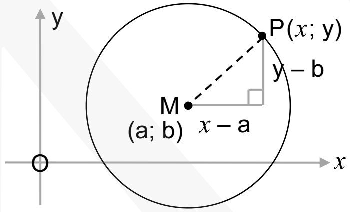


$$x^2 + y^2 = r^2$$

Thm. of Pythag.!

Circles with any given centre

True of any point **(x; y)** on a circle with centre **(a; b)** and radius **r** is that:



$$(x - a)^2 + (y - b)^2 = r^2$$

Distance formula! (Thm. of Pythag.)

Converting the equation of a circle

General form: $Ax^2 + Bx + Cy^2 + Dy + E = 0$

to **Standard form:** $(x - a)^2 + (y - b)^2 = r^2$

(using completion of squares)

e.g.

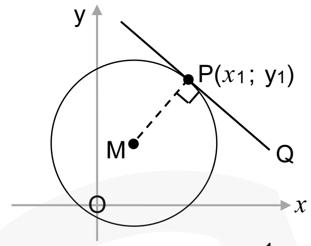
$$x^2 - 6x + y^2 + 8y - 25 = 0$$
$$\therefore x^2 - 6x + y^2 + 8y = 25$$
$$\therefore x^2 - 6x + \mathbf{3^2} + y^2 + 8y + \mathbf{4^2} = 25 + \mathbf{9} + \mathbf{16}$$
$$\therefore (x - 3)^2 + (y + 4)^2 = 50$$

This is the equation of a circle with:

centre $(3; -4)$ & **radius**, $r = \sqrt{50}\ (= 5\sqrt{2})$ units

A Tangent to a circle . . .

is **perpendicular** to the **radius** of the circle at the **point** of **contact**.



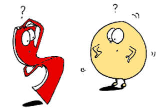
$m_{MP} = 2 \Rightarrow m_{PQ} = -\frac{1}{2}$

($\because$ radius MP $\perp$ tangent PQ)

To find the **equation** of a tangent, use **"m and 1 point"** in the straight line equation:

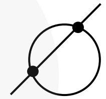
$$y - y_1 = m(x - x_1)$$

Point(s) of intersection of a Line and a Circle

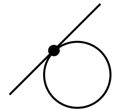


A line and a circle **either**

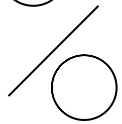
1. **"cut"** (twice!) [secant] (2 points in common)



or 2. **"touch"** (once!) [tangent] (1 point in common)



or 3. **don't cut or touch** (no points in common)



If we substitute $y = mx + c$ into the equation of the ⊙, there will **either** be: 1. **2 solutions** or 2. **1 solution** or 3. **no solutions** for x, resulting in one of the above scenarios.

FINAL ADVICE

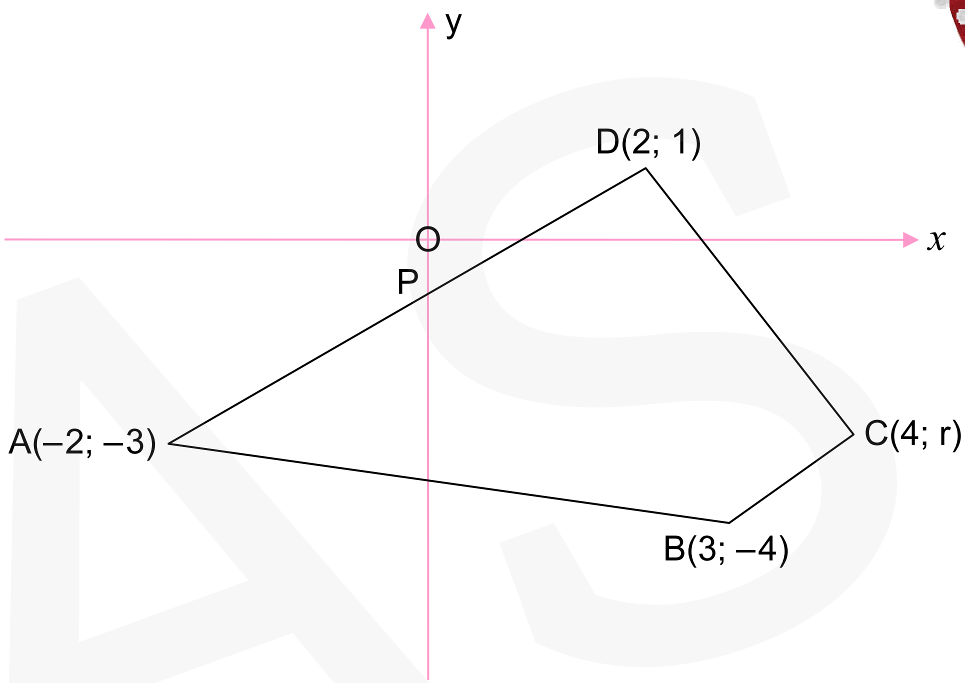
Use common sense

& ALWAYS DRAW A PICTURE!!!

SOME INTERESTING QUESTIONS

QUESTION 1

In the diagram alongside, points A(–2; –3), B(3; –4), C(4; r) and D(2; 1) are the vertices of quadrilateral ABCD.



P is the midpoint of line AD.

1.1 Calculate the value of r if AD || BC. (4)

1.2 What type of quadrilateral is ABCD? (1)

1.3 Determine the coordinates of P. (2)

1.4 Prove that BP ⊥ AD. (2)

1.5 Determine the equation of the circle passing through PBA in the form $(x - a)^2 + (y - b)^2 = r^2$. (5)

1.6 Calculate the maximum radius of the circle having equation $x^2 + y^2 - 2x\cos\theta - 4y\cos\theta = -2$ for any value of θ. (5)

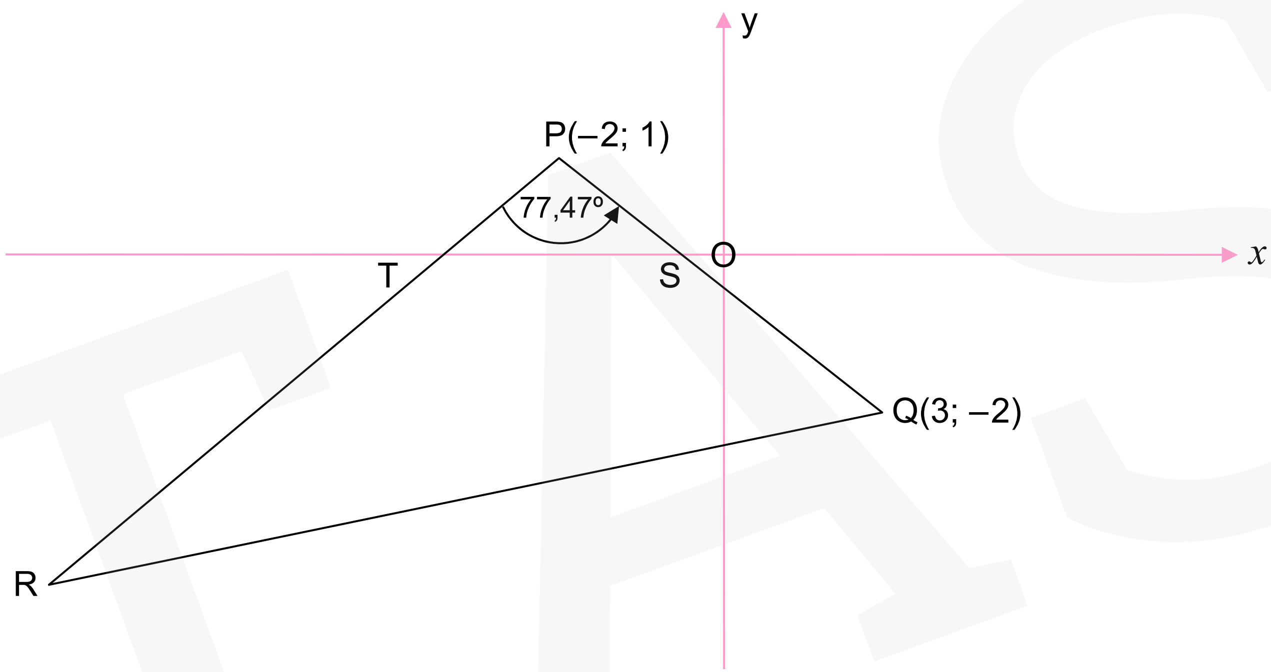
Gauteng 2018 P2 (Q3.1)

QUESTION 2

In the diagram below, points P(–2; 1) and Q(3; –2), are given and R is a point in the third quadrant.

PQ and PR cut the x-axis at S and T respectively.

$Q\hat{P}R = 77{,}47°$.



2.1 Determine the equation of line PQ in the form $ax + by + c = 0$ (3)

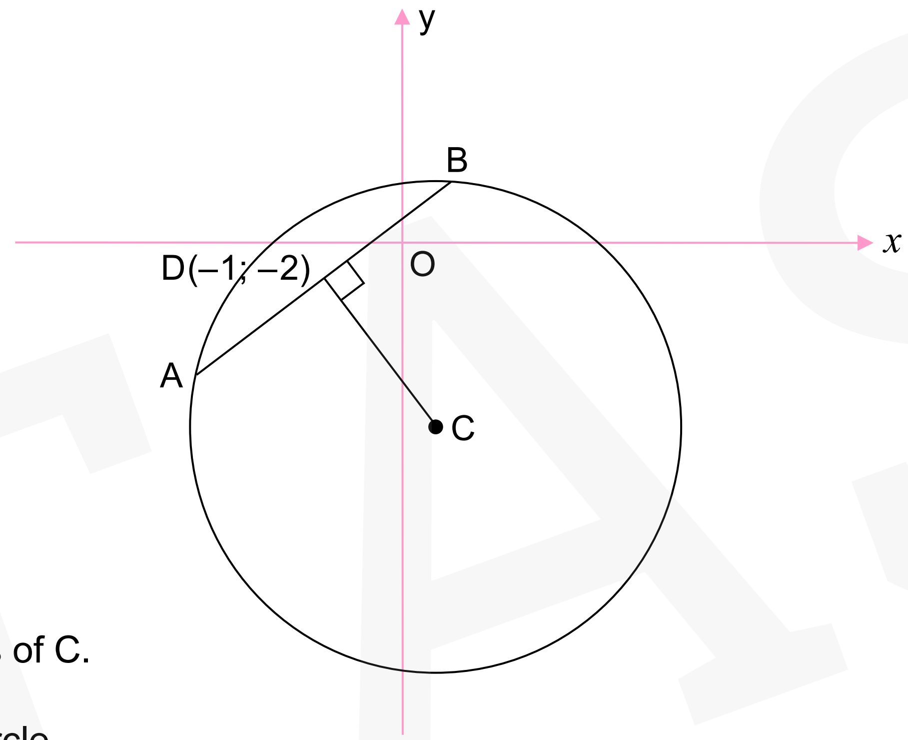
2.2 Determine the equation of PR in the form $y = mx + c$. (6)

Gauteng 2018 P2 (Q3.2)

QUESTION 3

In the diagram below, AB is a chord of the circle with centre C. D(–1; –2) is the midpoint of AB. DC ⊥ AB.

The equation of the circle is $x^2 + y^2 + 6y = 4x + 12$.



3.1 Determine the coordinates of C. (3)

3.2 Determine the radius of circle. (1)

3.3 Calculate the length of AB. (5)

3.4 Calculate the area of ΔABC. (3)

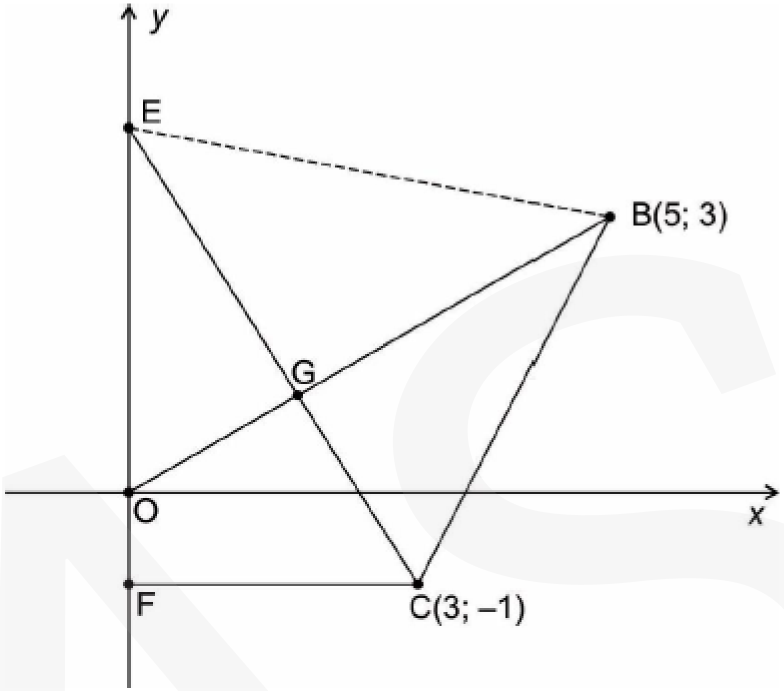
[12]

Gauteng 2018 P2 (Q4)

QUESTION 4

In the diagram alongside:

- E, O and F lie on the y-axis
- The coordinates of B(5; 3) and C(3; –1) are given
- EC intersects OB at G
- FC is parallel to the x-axis



4.1 Calculate the gradient of line OB. (1)

4.2 Determine the coordinates of E if EC is perpendicular to OB. (3)

4.3 Determine the length of straight line EB. (2)

4.4 Calculate the coordinates of point G. (5)

4.5.1 Calculate the area of $\triangle$EFC. (2)

4.5.2 Hence, or otherwise, calculate the area of the quadilateral OGCF. (3)

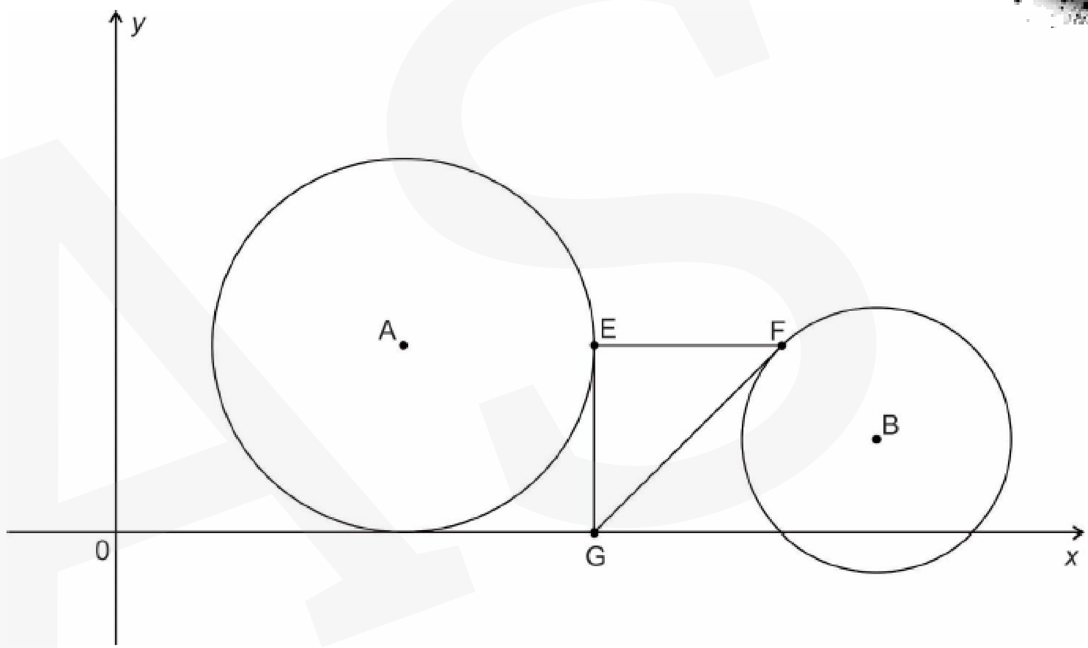
[16]

IEB May 2021 P2 (Q1)

QUESTION 5

In the diagram below:

- E and F lie on the circles with centre A and B respectively and EF is parallel to the x-axis
- G lies on the x-axis with GE a tangent to circle A and also perpendicular to the x-axis
- GF is a tangent to circle B at F
- Equation of circle A: $(x - 3)^2 + (y - 2)^2 = 4$
- Equation of circle B: $x^2 - 16x + y^2 - 2y + 63 = 0$



5.1 Determine the coordinates of point E. (3)

5.2 Determine the coordinates of B. (2)

5.2 Calculate the length of FG, leaving your answer in surd form. (6)

[11]

IEB May 2021 P2 (Q2)

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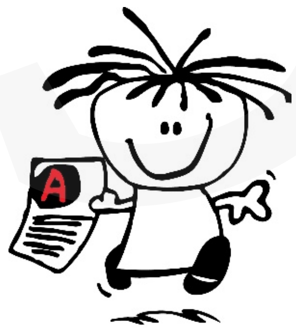
- include critical prior learning (Gr 10 & 11) when this foundation is required for mastering the entire FET curriculum
- engage learners eagerly as they participate and thrive on their maths journey
- accommodate all cognitive levels

The questions and detailed solutions have been provided in



SECTION 1: Separate topics

It is important that learners focus on and master one topic at a time BEFORE attempting 'past papers' which could be bewildering and demoralising. In this way they can develop confidence and a deep understanding.

SECTION 2: Exam Papers

When learners have worked through the topics and grown fluent, they can then move on to the exam papers to experience working through a variety of questions in one session, and to perfect their skills.

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Please submit your feedback by clicking on the link in the chat.

If you're having trouble finding the feedback form, please e-mail Jenny on **jenny@theanswerseries.co.za**

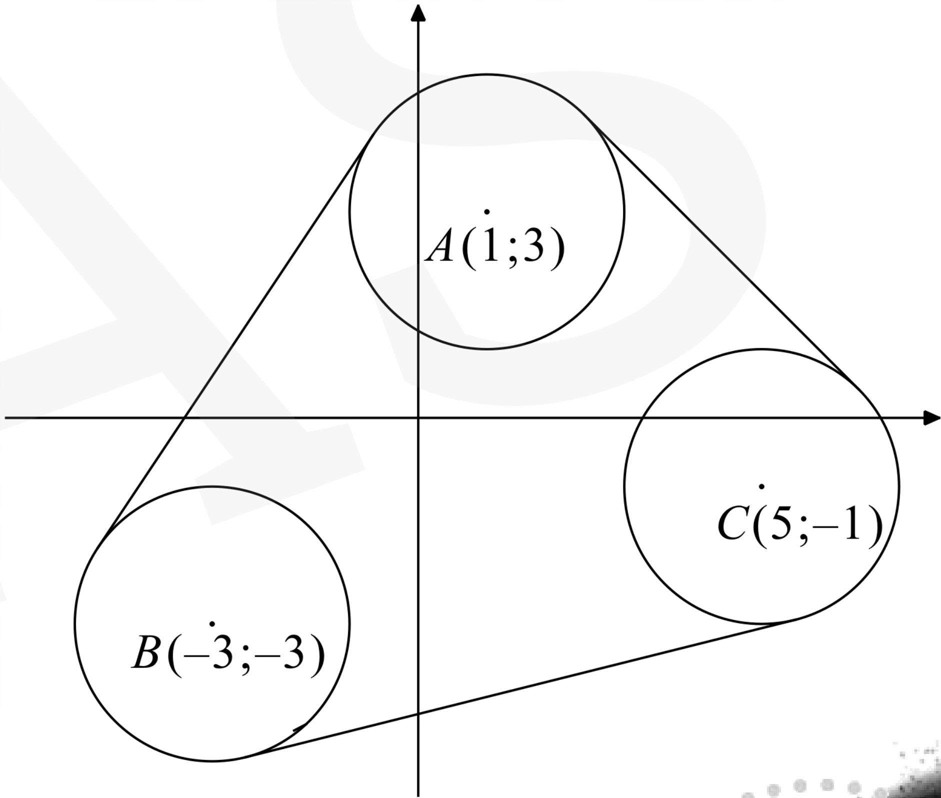
THANK YOU

Analytical Geometry Question 2

If you're having trouble filling in the feedback form, please e-mail Jenny on **jenny@theanswerseries.co.za**

Three circles, all with a radius of 2, have centres at A(1; 3); B(–3; –3) and C(5; –1).

A rubber band is stretched tightly around the circles.

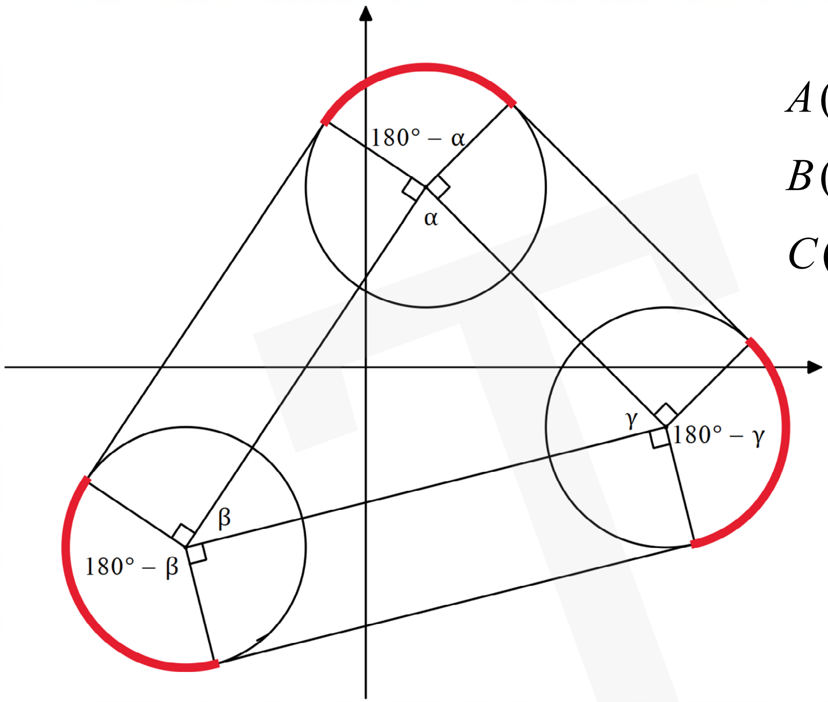


2.1 Determine the length of the rubber band.

2.2 Determine the area enclosed by the rubber band.

Analytical Geometry Question 2.1 Solution

Three circles, all with a radius of 2, have centres at A(1; 3); B(–3; –3) and C(5; –1). A rubber band is stretched tightly around the circles. Determine the length of the rubber band.



$A(1;3)$

$B(-3;-3)$

$C(5;-1)$

$AB = \sqrt{52}$; $BC = \sqrt{68}$; $AC = \sqrt{32}$

Note: $\alpha + \beta + \gamma = 180^\circ$ (∠ sum in Δ)

$$\therefore\ 180^\circ - \alpha + 180^\circ - \beta + 180^\circ - \gamma$$
$$= 540^\circ - (\alpha + \beta + \gamma)$$
$$= 540^\circ - 180^\circ$$
$$= 360^\circ$$

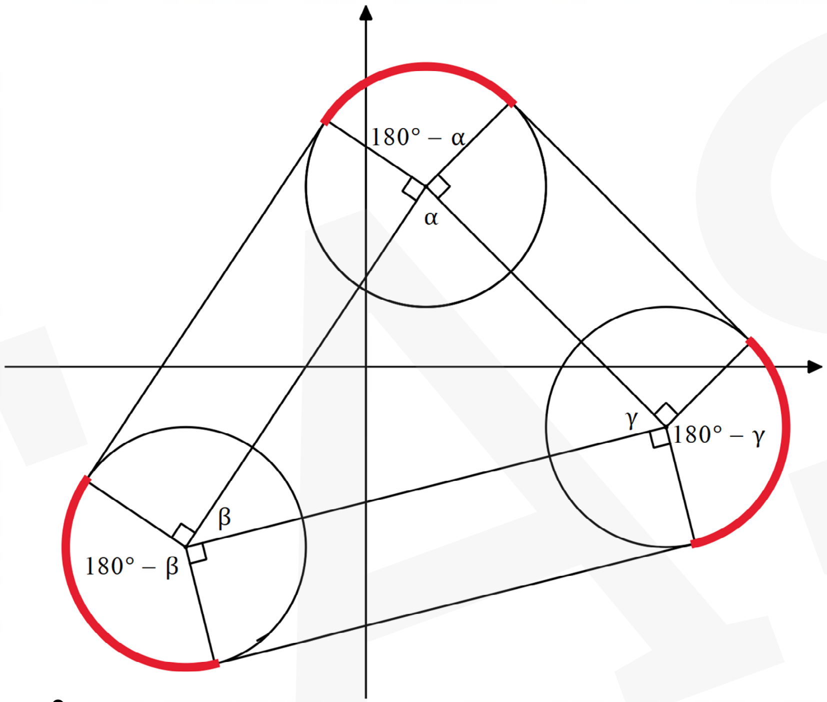
i.e. a full circle of radius 2

$$P = \sqrt{52} + \sqrt{68} + \sqrt{32} + 2\pi(2)$$

$$P = 33{,}68 \text{ units}$$

Analytical Geometry Question 2.2 Solution

Three circles, all with a radius of 2, have centres at A(1; 3); B(–3; –3) and C(5; –1).
A rubber band is stretched tightly around the circles. Determine the area enclosed by the rubber band.



$A(1; 3)$

$B(-3; -3)$

$C(5; -1)$

$AB = \sqrt{52}$

$BC = \sqrt{68}$

$AC = \sqrt{32}$

$$\cos\beta = \frac{(\sqrt{52})^2 + (\sqrt{68})^2 - (\sqrt{32})^2}{2 \times \sqrt{52} \times \sqrt{68}}$$

$$\therefore \beta = 42{,}27^\circ$$

$$\therefore A = \frac{1}{2} \times \sqrt{52} \times \sqrt{68} \times \sin 42{,}27^\circ + \sqrt{52} \times 2 + \sqrt{68} \times 2 + \sqrt{32} \times 2 + \pi \times 2^2$$

$$\therefore A = 20 + 14{,}42 + 16{,}49 + 11{,}31 + 12{,}57$$

$$\therefore A = 74{,}79 \text{ units}^2$$