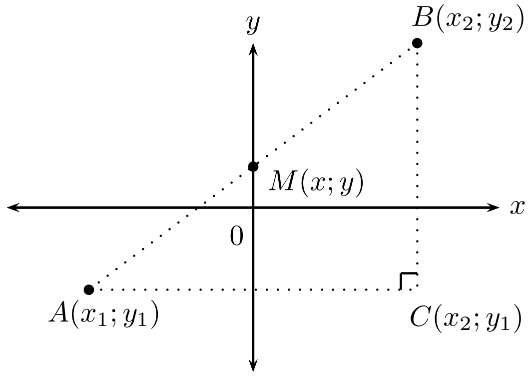
CHAPTER 7

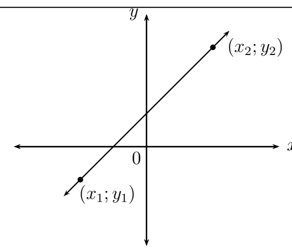
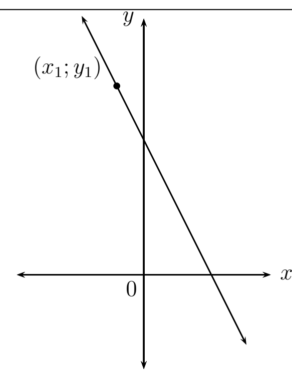
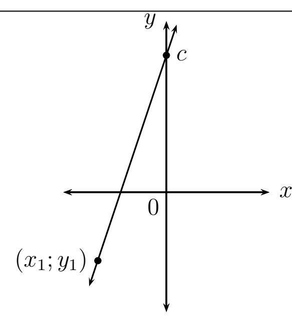
Analytical geometry

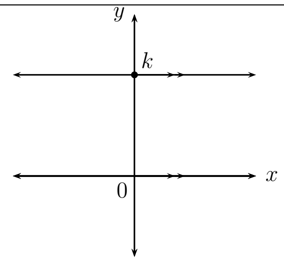
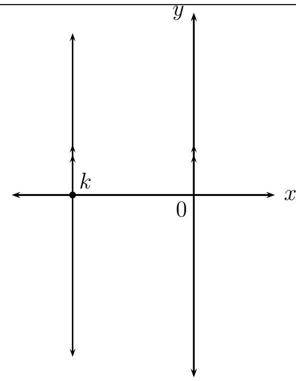
7.1 Revision EMCHN

Straight line equations EMCHP



Theorem of Pythagoras:	$AB^2 = AC^2 + BC^2$
Distance formula:	$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Gradient:	$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$ or $m_{AB} = \frac{y_1 - y_2}{x_1 - x_2}$
Mid-point of a line segment:	$M(x; y) = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$
Points on a straight line:	$m_{AB} = m_{AM} = m_{MB}$

Two-point form:	$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$	
Gradient-point form:	$y - y_1 = m(x - x_1)$	
Gradient-intercept form:	$y = mx + c$	

Horizontal lines:	$y = k$	
Vertical lines:	$x = k$	

Worked example 1: Revision

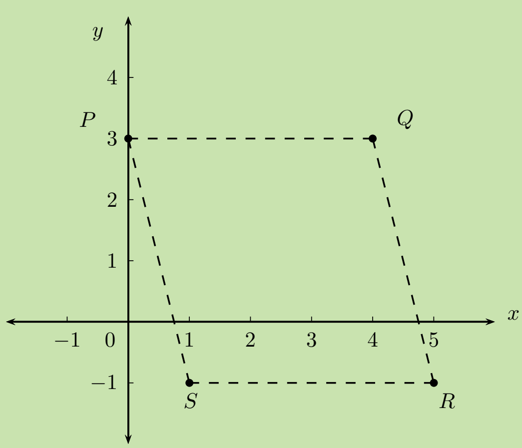
QUESTION

Given quadrilateral $PQRS$ with vertices $P(0;3), Q(4;3), R(5;-1)$ and $S(1;-1)$.

1. Determine the equation of the lines PS and QR.
2. Show that $PS \parallel QR$.
3. Calculate the lengths of PS and QR.
4. Determine the equation of the diagonal QS.
5. What type of quadrilateral is $PQRS$?

SOLUTION

Step 1: Draw a sketch



Step 2: Use the given information to determine the equation of lines PS and QR

$$\text{Gradient: } m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Two-point form: } \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Gradient-intercept form: } y = mx + c$$

Determine the equation of the line PS using the two point form of the straight line equation:

$$x_1 = 0; \qquad y_1 = 3; \qquad x_2 = 1; \qquad y_2 = -1$$

$$\begin{aligned}
\frac{y - y_1}{x - x_1} &= \frac{y_2 - y_1}{x_2 - x_1} \\
\frac{y - 3}{x - 0} &= \frac{-1 - 3}{1 - 0} \\
\frac{y - 3}{x} &= -4 \\
y - 3 &= -4x \\
\therefore y &= -4x + 3
\end{aligned}$$

Determine the equation of the line QR using the gradient-intercept form of the straight line equation:

$$\begin{aligned}
m_{QR} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{-1 - 3}{5 - 4} \\
&= \frac{-4}{1} \\
&= -4 \\
y &= mx + c \\
y &= -4x + c \\
\text{Substitute } (4;3) \quad 3 &= -4(4) + c \\
\therefore c &= 19 \\
y &= -4x + 19
\end{aligned}$$

There is often more than one method for determining the equation of a line. The different forms of the straight line equation are used, depending on the information provided in the problem.

Step 3: Show that line PS and line QR have equal gradients

$$\begin{aligned}
y &= -4x + 3 \\
\therefore m_{PS} &= -4 \\
\text{And } y &= -4x + 19 \\
\therefore m_{QR} &= -4 \\
\therefore m_{PS} &= m_{QR} \\
\therefore PS &\parallel QR
\end{aligned}$$

Step 4: Use the distance formula to determine the lengths of PS and QR

$$\begin{aligned} PS &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(1-0)^2 + (-1-3)^2} \\ &= \sqrt{1 + (-4)^2} \\ &= \sqrt{17} \text{ units} \end{aligned}$$

$$\begin{aligned} QR &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(5-4)^2 + (-1-3)^2} \\ &= \sqrt{1 + (-4)^2} \\ &= \sqrt{17} \text{ units} \end{aligned}$$

Step 5: Determine the equation of the diagonal QS

Determine the gradient of the line:

$$\begin{aligned} m_{QS} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-1-3}{1-4} \\ &= \frac{-4}{-3} \\ &= \frac{4}{3} \end{aligned}$$

Use gradient and the point $Q(4;3)$ to determine the equation of the line QS:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - y_1 &= \frac{4}{3}(x - x_1) \\ \text{Substitute } (4;3) \quad y - 3 &= \frac{4}{3}(x - 4) \\ y - 3 &= \frac{4}{3}x - \frac{16}{3} \\ y &= \frac{4}{3}x - \frac{16}{3} + 3 \\ \therefore y &= \frac{4}{3}x - \frac{7}{3} \end{aligned}$$

Step 6: Examine the properties of quadrilateral $PQRS$

We have shown that $PS \parallel QR$ and $PS = QR$, therefore quadrilateral $PQRS$ is a parallelogram (one pair of opposite sides equal and parallel).

Exercise 7 – 1: Revision

1. Determine the following for the line segment between the given points:

   - length
   - mid-point
   - gradient
   - equation

   a) $(-2;-4)$ and $(3;11)$

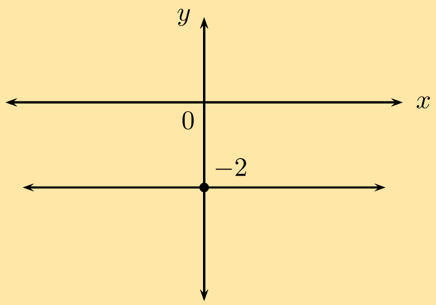
   b) $(-5;-3)$ and $(10;6)$

   c) $(h;-h-k)$ and $(2k;h-5k)$

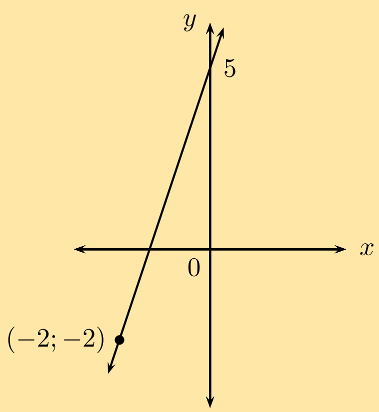
   d) $(2;9)$ and $(0;-1)$

2. The line joining $A(x;y)$ and $B(-3;6)$ has the mid-point $M(2;3)$. Determine the values of x and y.
3. Given $F(2;11), G(-4;r)$ and length $FG = 6\sqrt{5}$ units, determine the value(s) of r.
4. Determine the equation of the straight line:
   a) passing through the point $(\frac{1}{2};4)$ and $(1;5)$.
   b) passing through the points $(2;-3)$ and $(-1;0)$.
   c) passing through the point $(9;1)$ and with $m = \frac{1}{3}$.
   d) parallel to the x-axis and passing through the point $(0;-4)$.
   e) passing through the point $(\frac{1}{2};-1)$ and with $m = -4$.
   f) perpendicular to the x-axis and passing through the point $(5;0)$.
   g) with undefined gradient and passing through the point $(\frac{3}{4};0)$.
   h) with $m = 2p$ and passing through the point $(3;6p+3)$.
   i) which cuts the y-axis at $y = -\frac{3}{5}$ and with $m = 4$.

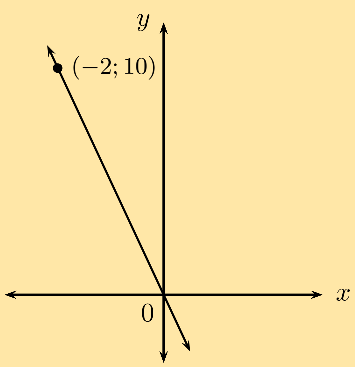
   j)



   k)



   l)



5. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 293J	1b. 293K	1c. 293M	1d. 293N	2. 293P	3. 293Q
4a. 293R	4b. 293S	4c. 293T	4d. 293V	4e. 293W	4f. 293X
4g. 293Y	4h. 293Z	4i. 2942	4j. 2943	4k. 2944	4l. 2945

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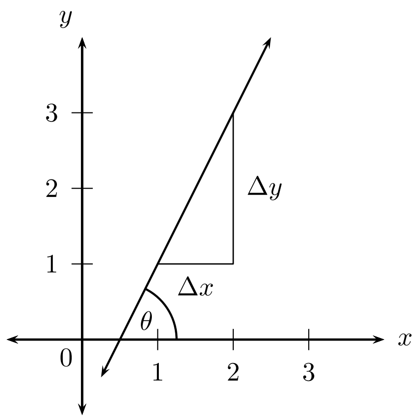
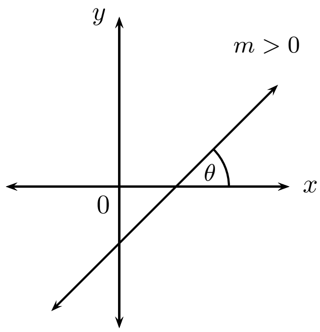


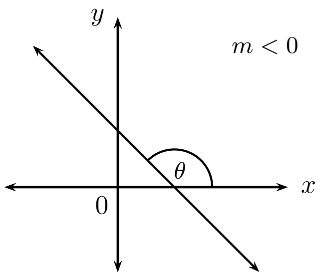
The diagram shows a straight line which forms an acute angle θ with the positive x-axis. This is called the **angle of inclination** of a straight line.

The gradient of a straight line is equal to the tangent of the angle formed between the line and the positive direction of the x-axis.

$$m = \tan\theta \qquad \text{for } 0° \leq \theta < 180°$$



θ is an acute angle



θ is an obtuse angle

Lines with positive gradients

A line with a positive gradient ($m > 0$) has an acute angle of inclination ($0° < \theta < 90°$).

For example, we can determine the angle of inclination of a line with $m = 1{,}2$:

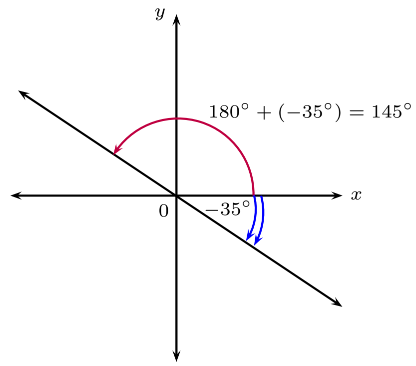
$$\begin{aligned} \tan\theta &= m \\ &= 1{,}2 \\ \therefore \theta &= \tan^{-1}(1{,}2) \\ &= 50{,}2° \end{aligned}$$

Lines with negative gradients

If we are calculating the angle of inclination of a line with a negative gradient ($m < 0$), then we add $180°$ to change the negative angle to an obtuse angle ($90° < \theta < 180°$).

For example, we can determine the angle of inclination for a line with $m = -0{,}7$:

$$\begin{aligned} \tan\theta &= m \\ &= -0{,}7 \\ \therefore \theta &= \tan^{-1}(-0{,}7) \\ &= -35{,}0° \\ \text{Obtuse angle: } \theta &= -35{,}0° + 180° \\ &= 145° \end{aligned}$$



Worked example 2: Inclination of a straight line

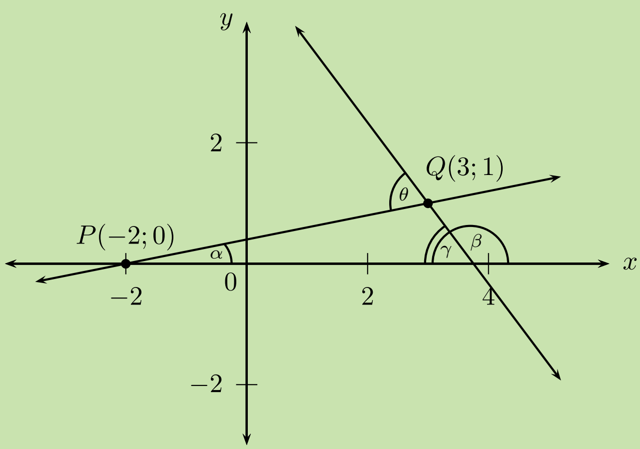
QUESTION

Determine the acute angle (correct to 1 decimal place) between the line passing through the points $P(-2;0)$ and $Q(3;1)$ and the straight line $y = -\frac{4}{3}x + 5$.

SOLUTION

Step 1: Draw a sketch

Draw the line through points $P(-2;0)$ and $Q(3;1)$ and the line $y = -\frac{4}{3}x + 5$ on a suitable system of axes. Label α and β, the angles of inclination of the two lines. Label θ, the acute angle between the two straight lines.



Notice that α and θ are acute angles and β is an obtuse angle.

$$\gamma = 180° - \beta \qquad (\angle \text{ on str. line})$$

$$\text{and } \theta = \alpha + \gamma \qquad (\text{ext. } \angle \text{ of } \triangle = \text{ sum int. opp})$$

$$\therefore \theta = \alpha + (180° - \beta)$$

$$= 180° + \alpha - \beta$$

Step 2: Use the gradient to determine the angle of inclination β

From the equation $y = -\frac{4}{3}x + 5$ we see that $m < 0$, therefore β is an obtuse angle.

$$\begin{aligned} m &= -\frac{4}{3} \\ \tan\beta &= -\frac{4}{3} \\ \therefore \beta &= \tan^{-1}\left(-\frac{4}{3}\right) \\ &= -53{,}1° \\ \beta &= -53{,}1° + 180° \\ &= 126{,}9° \end{aligned}$$

Step 3: Determine the gradient and angle of inclination of the line through P and Q

Determine the gradient

$$\begin{aligned} m &= \frac{y_P - y_Q}{x_P - x_Q} \\ &= \frac{-1}{-5} \\ &= \frac{1}{5} \end{aligned}$$

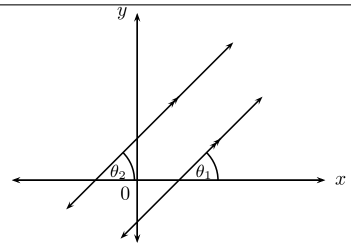
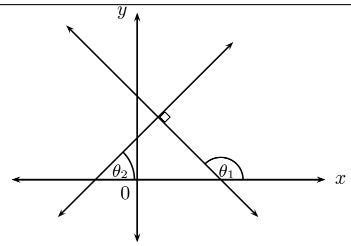
Determine the angle of inclination

$$\begin{aligned} \tan\alpha &= m \\ &= \frac{1}{5} \\ \therefore \alpha &= \tan^{-1}\left(\frac{1}{5}\right) \\ &= 11{,}3° \end{aligned}$$

Step 4: Write the final answer

$$\begin{aligned} \theta &= 180° + \alpha - \beta \\ &= 180° + 11{,}3° - 126{,}9° \\ &= 64{,}4° \end{aligned}$$

The acute angle between the two straight lines is $64{,}4°$.

Parallel lines		$m_1 = m_2$	$\theta_1 = \theta_2$
Perpendicular lines		$m_1 \times m_2 = -1$	$\theta_1 = 90° + \theta_2$

Worked example 3: Parallel lines

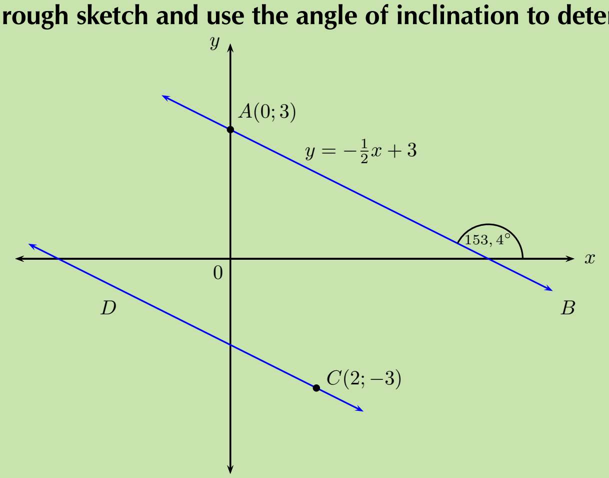
QUESTION

Line AB passes through the point $A(0;3)$ and has an angle of inclination of $153{,}4°$.

1. Determine the equation of line CD which passes through the point $C(2;-3)$ and is parallel to AB.
2. Determine the equation of line EF, which passes through the origin and is perpendicular to both AB and CD.
3. Sketch lines AB, CD and EF on the same system of axes.
4. Use two different methods to determine the angle of inclination of EF.

SOLUTION

Step 1: Draw a rough sketch and use the angle of inclination to determine the equation of CD



$$\begin{aligned} m_{AB} &= \tan\theta \\ &= \tan 153{,}4° \\ &= -0{,}5 \end{aligned}$$

Since we are given $AB \parallel CD$,

$$m_{CD} = m_{AB} = -0{,}5 = -\frac{1}{2}$$
$$y - y_1 = m(x - x_1)$$
$$y - y_1 = -\frac{1}{2}(x - x_1)$$

Substitute the given point $(2; -3)$:

$$y - (-3) = -\frac{1}{2}(x - 2)$$
$$y + 3 = -\frac{1}{2}x + 1$$
$$y = -\frac{1}{2}x - 2$$

Step 2: Determine the equation of EF

EF is perpendicular to AB, therefore the product of their gradients is equal to -1:

$$m_{AB} \times m_{EF} = -1$$
$$-\frac{1}{2} \times m_{EF} = -1$$
$$\therefore m_{EF} = 2$$

We know line EF passes through $(0; 0)$, therefore the equation of the line is:

$$y = 2x$$

Step 3: Determine the angle of inclination of EF

Let the angle of inclination of EF be β.

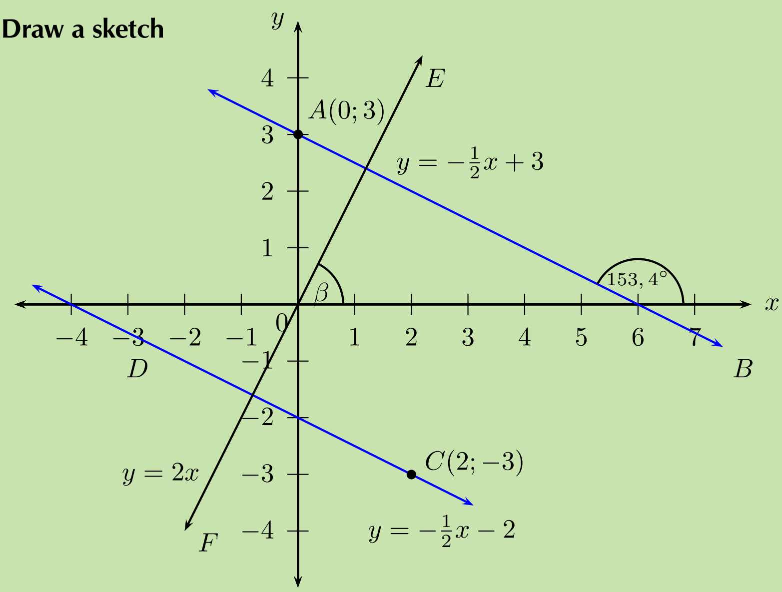
Method 1:

$$\beta = 153{,}4^\circ - 90^\circ$$
$$= 63{,}4^\circ$$

Method 2:

$$m = 2$$
$$\tan \beta = 2$$
$$\therefore \beta = 63{,}4^\circ$$

Step 4: Draw a sketch



Exercise 7 – 2: Inclination of a straight line

1. Determine the angle of inclination (correct to 1 decimal place) for each of the following:

   a) a line with $m = \frac{3}{4}$

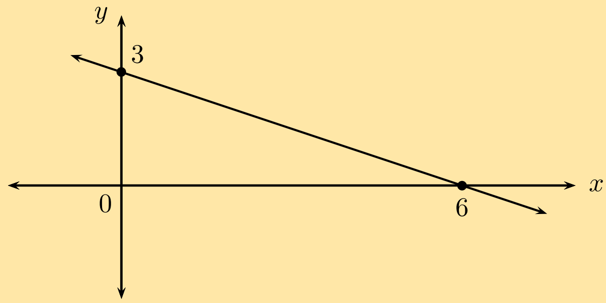
   b) $6 + x = 2y$

   c) the line passes through the points $(-4; 0)$ and $(2; 6)$

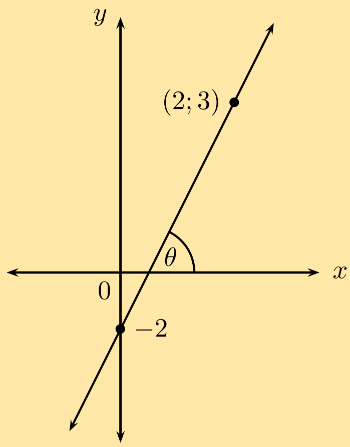
   d) $y = 4$

   e) a line with a gradient of $1{,}733$

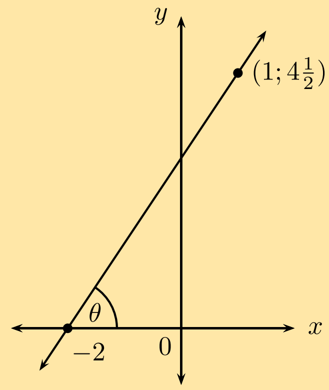
   f)



   g)



   h)



2. Find the angle between the line $2y = 5x$ and the line passing through points $T(2; 1\frac{1}{3})$ and V$(-3; 3)$.
3. Determine the equation of the straight line that passes through the point $(1; 2)$ and is parallel to the line $y + 3x = 1$.
4. Determine the equation of the straight line that passes through the point $(-4; -4)$ and is parallel to the line with angle of inclination $\theta = 56{,}31°$.
5. Determine the equation of the straight line that passes through the point $(1; -6)$ and is perpendicular to the line $5y = x$.
6. Determine the equation of the straight line that passes through the point $(3; -1)$ and is perpendicular to the line with angle of inclination $\theta = 135°$.
7. $A(2; 3)$, $B(-4; 0)$ and $C(5; -3)$ are the vertices of $\triangle ABC$ in the Cartesian plane. AC intersects the x-axis at D. Draw a sketch and determine the following:

   a) the equation of line AC

   b) the coordinates of point D

c) the angle of inclination of AC

d) the gradient of line AB

e) $B\hat{A}C$

f) the equation of the line perpendicular to AB and passing through the origin

g) the mid-point M of BC

h) the equation of the line parallel to AC and passing through point M

8. Points $F(-3;5)$, $G(-7;-4)$ and $H(2;0)$ are given.

   a) Plot the points on the Cartesian plane.

   b) Determine the coordinates of I if $FGHI$ is a parallelogram.

   c) Prove that $FGHI$ is a rhombus.

9. Given points $S(2;5)$, $T(-3;-4)$ and $V(4;-2)$.

   a) Show that the equation of the line ST is $5y = 9x + 7$.

   b) Determine the size of $T\hat{S}V$.

10. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 2946	1b. 2947	1c. 2948	1d. 2949	1e. 294B	1f. 294C
1g. 294D	1h. 294F	2. 294G	3. 294H	4. 294J	5. 294K
6. 294M	7. 294N	8. 294P	9. 294Q		

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7.2 Equation of a circle EMCHS

Equation of a circle with centre at the origin EMCHT

Investigation:

1. Draw a system of axes with a scale of 1 cm $=$ 1 unit on the x-axis and on the y-axis.
2. Draw the lines $y = x$ and $y = -x$.
3. Plot the following points:

   a) $O(0;0)$

   b) $D(2;0)$

   c) $E(\sqrt{2};\sqrt{2})$

   d) $F(0;2)$

   e) $G(-\sqrt{2};\sqrt{2})$

   f) $H(-2;0)$

   g) $I(-\sqrt{2};-\sqrt{2})$

   h) $J(0;-2)$

   i) $K(\sqrt{2};-\sqrt{2})$

   What object do the points form?

4. Measure the following distances and complete the table below:

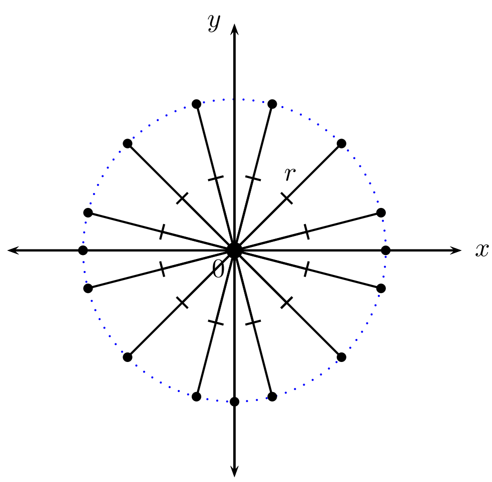
Line segment	Distance (cm)
DO	
EO	
FO	
GO	
HO	
IO	
JO	
KO	

5. Use the distance formula to check the results of the above table:

$$\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Line segment	Distance
DO	
EO	
FO	
GO	
HO	
IO	
JO	
KO	

6. What do you notice about the length of each line segment?
7. What is the general term given to this type of line segment?
8. If the point $P(x; y)$ lies on the circle, use the distance formula to determine an expression for the length of PO.
9. Can you deduce a general equation for a circle with centre at the origin?

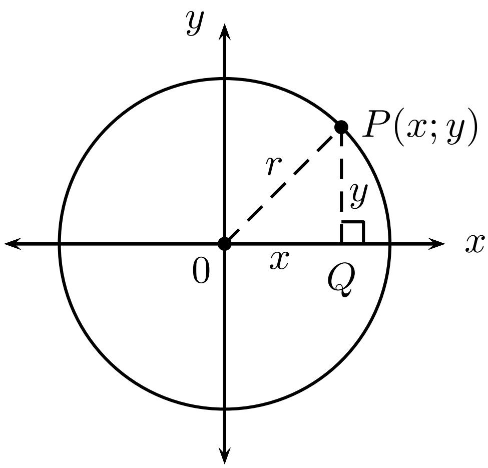


A circle is the set of all points that are an equal distance (radius) from a given point (centre). In other words, every point on the circumference of a circle is equidistant from its centre.

The radius of a circle is the distance from the centre of a circle to any point on the circumference.

A diameter of a circle is any line passing through the centre of the circle which connects two points on the circle. The diameter is also the name given to the maximum distance between two points on a circle.

Consider a point $P(x;y)$ on the circumference of a circle of radius r with centre at $O(0;0)$.



In $\triangle OPQ$:

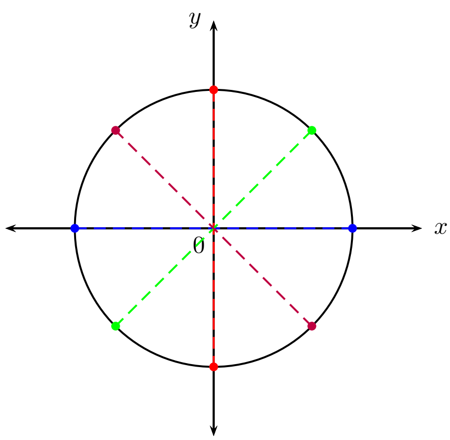
$$OP^2 = PQ^2 + OQ^2 \quad \text{(Pythagoras)}$$
$$OP = r$$
$$PQ = y - 0$$
$$OQ = x - 0$$
$$r^2 = (y-0)^2 + (x-0)^2$$
$$\therefore r^2 = x^2 + y^2$$

Equation of a circle with centre at the origin:
If $P(x;y)$ is a point on a circle with centre $O(0;0)$ and radius r, then the equation of the circle is:

$$x^2 + y^2 = r^2$$

Circle symmetry

A circle with centre $(0;0)$ is symmetrical about the origin: for every point $(x;y)$ on the circumference of a circle, there is also the point $(-x;-y)$.



A circle centred on the origin is also symmetrical about the x- and y-axis. Is a circle centred on the origin symmetrical about the lines $y=x$ and $y=-x$? How many lines of symmetry does a circle have?

Worked example 4: Equation of a circle with centre at the origin

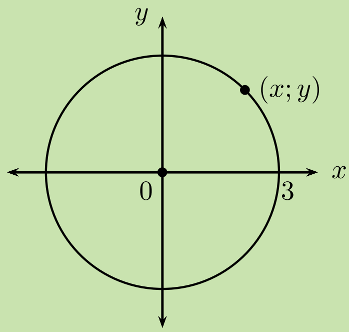
QUESTION

Given: circle with centre $O(0;0)$ and a radius of 3 units.

1. Sketch the circle on the Cartesian plane.
2. Determine the equation of the circle.
3. Show that the point $T\left(-\sqrt{4};\sqrt{5}\right)$ lies on the circle.

SOLUTION

Step 1: Draw a sketch



Step 2: Determine the equation of the circle

Write down the general form of the equation of a circle with centre $(0;0)$:

$$\begin{aligned} & x^2+y^2=r^2 \\ \text{Substitute } r=3: \quad & x^2+y^2=(3)^2 \\ & x^2+y^2=9 \end{aligned}$$

Step 3: Show that point T lies on the circle

Substitute the x-coordinate and the y-coordinate into the left-hand side of the equation and show that it is equal to the right-hand side:

$$\begin{aligned} \text{LHS} &= x^2+y^2 \\ &= \left(-\sqrt{4}\right)^2+\left(\sqrt{5}\right)^2 \\ &= 4+5 \\ &= 9 \\ &= r^2 \\ &= \text{RHS} \end{aligned}$$

Therefore, $T\left(-\sqrt{4};\sqrt{5}\right)$ lies on the circle $x^2+y^2=9$.

Worked example 5: Equation of a circle with centre at the origin

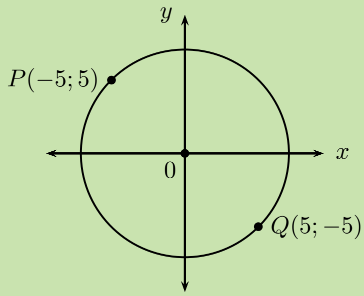
QUESTION

A circle with centre $O(0;0)$ passes through the points $P(-5;5)$ and $Q(5;-5)$.

1. Plot the points and draw a rough sketch of the circle.
2. Determine the equation of the circle.
3. Calculate the length of PQ.
4. Explain why PQ is a diameter of the circle.

SOLUTION

Step 1: Draw a sketch



Step 2: Determine the equation of the circle

Write down the general form of the equation of a circle with centre $(0;0)$ and substitute $P(-5;5)$:

$$\begin{aligned} x^2 + y^2 &= r^2 \\ (-5)^2 + (5)^2 &= r^2 \\ 25 + 25 &= r^2 \\ 50 &= r^2 \\ \therefore r &= \sqrt{50} \quad (r \text{ is always positive}) \\ r &= 5\sqrt{2} \text{ units} \end{aligned}$$

Therefore, the equation of the circle passing through P and Q is $x^2 + y^2 = 50$.

Step 3: Calculate the length PQ

Use the distance formula to determine the distance between the two points.

$$\begin{aligned} PQ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(5 - (-5))^2 + (-5 - 5)^2} \\ &= \sqrt{(10)^2 + (-10)^2} \\ &= \sqrt{100 + 100} \\ &= \sqrt{100 \;.\; 2} \\ &= 10\sqrt{2} \text{ units} \end{aligned}$$

Step 4: Determine if PQ is a diameter of the circle

$$\begin{aligned} r &= 5\sqrt{2} \\ \text{And } d &= 2 \times r \\ &= 2 \times 5\sqrt{2} \\ &= 10\sqrt{2} \\ \therefore d &= PQ \end{aligned}$$

Since PQ connects two points, P and Q, on the circle and is a line of length $10\sqrt{2}$ units, PQ is a diameter of the circle.

Alternative method 1: Use symmetry to show that PQ is a diameter of the circle.

The diameter is the name given to the maximum distance between two points on a circle; this means that the two points must lie opposite each other with respect to the centre.

Using symmetry about the origin, we know that $(x; y)$ lies opposite $(-x; -y)$ on the circle and vice versa:

- $P(-5; 5)$ lies opposite $(5; -5)$, which are the coordinates of Q
- $Q(5; -5)$ lies opposite $(-5; 5)$, which are the coordinates of P

Therefore, PQ is a diameter of the circle.

Alternative method 2: Show that PQ passes through the centre.

Determine the equation of PQ and show that it passes through the origin.

$$\begin{aligned} \frac{y - y_Q}{x - x_Q} &= \frac{y_P - y_Q}{x_P - x_Q} \\ \frac{y + 5}{x - 5} &= \frac{5 + 5}{-5 - 5} \\ \frac{y + 5}{x - 5} &= \frac{10}{-10} \\ y + 5 &= -(x - 5) \\ y &= -x + 5 - 5 \\ y &= -x \end{aligned}$$

$\therefore (0; 0)$ lines on the line PQ.

Therefore, PQ passes through the centre and is a diameter of the circle.

Worked example 6: Equation of a circle with centre at the origin

QUESTION

Given a circle with centre $O(0;0)$ and a radius of $\sqrt{45}$ units. Determine the possible coordinates of the point(s) on the circle which have an x-value that is twice the y-value.

SOLUTION

Step 1: Determine the equation of the circle

$$\begin{aligned} x^2 + y^2 &= r^2 \\ x^2 + y^2 &= \left(\sqrt{45}\right)^2 \\ x^2 + y^2 &= 45 \end{aligned}$$

Step 2: Determine the coordinates of the points on the circle

To calculate the possible coordinates of the point(s) on the circle which have an x-value that is twice the y-value, we substitute $x = 2y$ into the equation of the circle:

$$\begin{aligned} x^2 + y^2 &= 45 \\ (2y)^2 + y^2 &= 45 \\ 4y^2 + y^2 &= 45 \\ 5y^2 &= 45 \\ y^2 &= 9 \\ \therefore y &= \pm 3 \end{aligned}$$

$$\begin{aligned} \text{If } y = -3: \quad x &= 2(-3) \\ &= -6 \\ \text{If } y = 3: \quad x &= 2(3) \\ &= 6 \end{aligned}$$

This gives the points $(6;3)$ and $(-6;-3)$.

Note: we can check that both points lie on the circle by substituting the coordinates into the equation of the circle:

$$\begin{aligned} (6)^2 + (3)^2 &= 36 + 9 = 45 \\ (-6)^2 + (-3)^2 &= 36 + 9 = 45 \end{aligned}$$

Exercise 7 – 3: Equation of a circle with centre at the origin

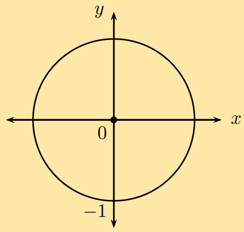
1. Complete the following for each circle given below:
   - Determine the radius.
   - Draw a sketch.
   - Calculate the coordinates of two points on the circle.

   a) $x^2 + y^2 = 16$
   b) $x^2 + y^2 = 100$
   c) $3x^2 + 3y^2 = 27$
   d) $y^2 = 20 - x^2$
   e) $x^2 + y^2 = 2{,}25$
   f) $y^2 = -x^2 + \frac{10}{9}$

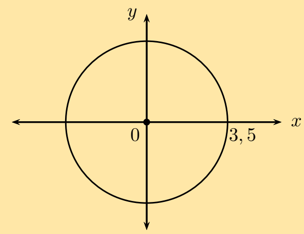
2. Determine the equation of the circle:

   a) with centre at the origin and a radius of 5 units.
   b) with centre at $(0;0)$ and $r = \sqrt{11}$ units.
   c) passing through the point $(3;5)$ and with centre $(0;0)$.
   d) centred at the origin and $r = 2{,}5$ units.
   e) with centre at the origin and a diameter of 30 units.
   f) passing through the point $(p;3q)$ and with centre at the origin.

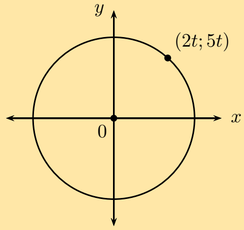
   g)



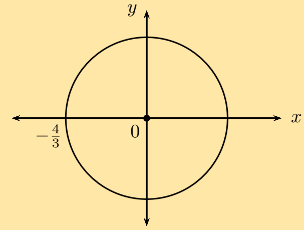
   i)



   h)



   j)



3. Determine whether or not the following equations represent a circle:

   a) $x^2 + y^2 - 8 = 0$
   b) $y^2 - x^2 + 25 = 0$
   c) $3x^2 + 6y^2 = 18$
   d) $x^2 = \sqrt{6} - y^2$
   e) $y(y + x) = -x(x - y) + 11$
   f) $\sqrt{80} + x^2 - y^2 = 0$
   g) $\frac{y^2}{3} + \frac{x^2}{3} = 3$

4. Determine the value(s) of g if $\left(\sqrt{3};g\right)$ is a point on the circle $x^2 + y^2 = 19$.
5. $A(s;t)$ is a point on the circle with centre at the origin and a diameter of 40 cm.

   a) Determine the possible coordinates of A if the value of s is triple the value of t.
   b) Determine the possible coordinates of A if the value of s is half the value of t.

6. $P(-2;3)$ lies on a circle with centre at $(0;0)$.

   a) Determine the equation of the circle.
   b) Sketch the circle and label point P.
   c) If PQ is a diameter of the circle, determine the coordinates of Q.
   d) Calculate the length of PQ.
   e) Determine the equation of the line PQ.
   f) Determine the equation of the line perpendicular to PQ and passing through the point P.

7. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 294S 1b. 294T 1c. 294V 1d. 294W 1e. 294X 1f. 294Y
2a. 294Z 2b. 2952 2c. 2953 2d. 2954 2e. 2955 2f. 2956
2g. 2957 2h. 2958 2i. 2959 2j. 295B 3a. 295C 3b. 295D
3c. 295F 3d. 295G 3e. 295H 3f. 295J 3g. 295K 4. 295M
5. 295N 6. 295P

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Equation of a circle with centre at $(a; b)$ EMCHV

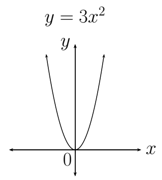
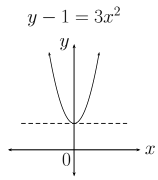
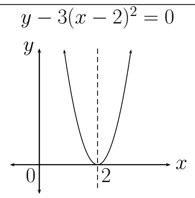
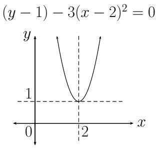
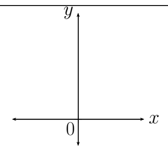
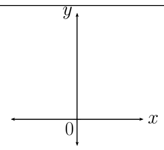
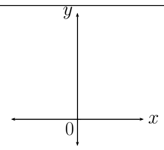
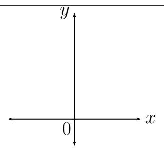
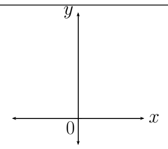
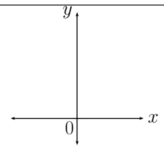
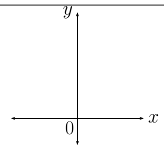
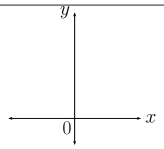
Investigation: Shifting the centre of a circle

Complete the following for each equation in the table below:

- Write down the shifted equation (do not simplify).
- Draw a rough sketch to illustrate the shift(s).

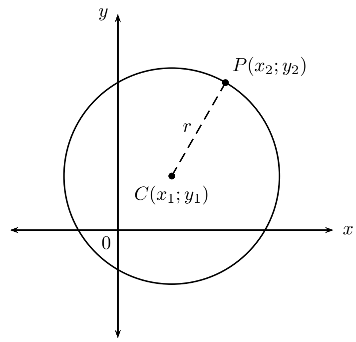
1. Vertical shift: the graph is shifted 1 unit up.
2. Horizontal shift: the graph is shifted 2 units to the right.
3. Combined shifts: the graph is shifted 1 unit up and 2 units to the right.

The first example has been completed.

Equation	Vertical shift	Horizontal shift	Combined shift
$y - 3x^2 = 0$	$(y-1) - 3x^2 = 0$	$y - 3(x-2)^2 = 0$	$(y-1) - 3(x-2)^2 = 0$
			
$y - 5^x = 0$			
			
$x^2 + y^2 = 4$			
			

Use the table to answer the following questions:

1. Write down the general equation of a circle with centre $(0; 0)$.
2. Write down the general equation of a circle with centre $(0; b)$.
3. Write down the general equation of a circle with centre $(a; 0)$.
4. Write down the general equation of a circle with centre $(a; b)$.



Consider a circle in the Cartesian plane with centre at $C(x_1; y_1)$ and with a radius of r units. If $P(x_2; y_2)$ is any point on the circumference of the circle, we can use the distance formula to calculate the distance between the two points:

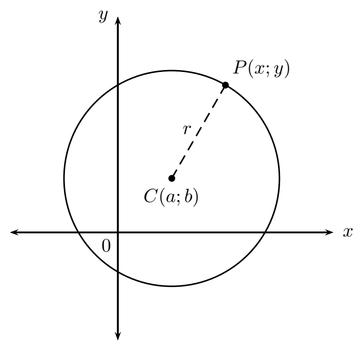
$$PC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The distance PC is equal to the radius (r) of the circle:

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$\therefore r^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

If the coordinates of the centre of the circle are $(a; b)$, then the equation of a circle not centred on the origin is:

$$(x_2 - a)^2 + (y_2 - b)^2 = r^2$$



Equation of a circle with centre at $(a; b)$:
If $P(x; y)$ is a point on a circle with centre $C(a; b)$ and radius r, then the equation of the circle is:

$$(x - a)^2 + (y - b)^2 = r^2$$

A circle with centre $(0; 0)$ is a special case of the general equation:

$$(x - a)^2 + (y - b)^2 = r^2$$

$$(x - 0)^2 + (y - 0)^2 = r^2$$
$$\therefore x^2 + y^2 = r^2$$

Worked example 7: Equation of a circle with centre at $(a;b)$

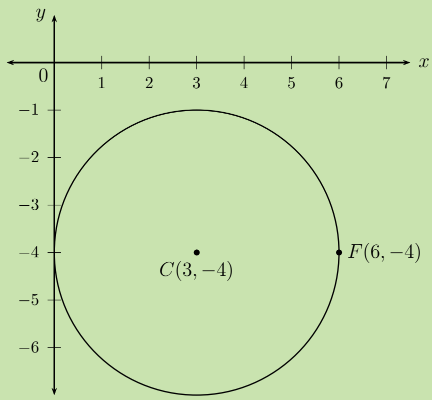
QUESTION

$F(6;-4)$ is a point on the circle with centre $(3;-4)$.

1. Draw a rough sketch of the circle and label F.
2. Determine the equation of the circle.
3. Does the point $G\left(\frac{3}{2};-2\right)$ lie on the circle?
4. Does the circle cut the y-axis? Motivate your answer.

SOLUTION

Step 1: Draw a sketch



Step 2: Determine the equation of the circle

Write down the general equation of a circle with centre $(a;b)$ and substitute the coordinates $(3;-4)$:

$$\begin{aligned}(x-a)^2+(y-b)^2&=r^2\\(x-3)^2+(y-(-4))^2&=r^2\\(x-3)^2+(y+4)^2&=r^2\end{aligned}$$

Substitute the coordinates of $F(6;-4)$ to determine the value of r^2:

$$\begin{aligned}(6-3)^2+(-4+4)^2&=r^2\\(3)^2+(0)^2&=r^2\\9&=r^2\\\therefore (x-3)^2+(y+4)^2&=9\end{aligned}$$

Step 3: Determine whether or not G lies on the circle

If $G\left(\frac{3}{2}; -2\right)$ lies on the circle, then it will satisfy the equation of the circle:

$$\begin{aligned}
\text{LHS} &= \left(\frac{3}{2} - 3\right)^2 + (-2+4)^2 \\
&= \left(-\frac{3}{2}\right)^2 + (2)^2 \\
&= \frac{9}{4} + 4 \\
&= \frac{9}{4} + \frac{16}{4} \\
&= \frac{25}{4}
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= 9 \\
\therefore \text{LHS} &\neq \text{RHS}
\end{aligned}$$

Therefore G does not lie on the circle.

Step 4: Determine the y-intercept(s)

To determine the y-intercept(s), we let $x = 0$:

$$\begin{aligned}
(0-3)^2 + (y+4)^2 &= 9 \\
9 + (y+4)^2 &= 9 \\
(y+4)^2 &= 0 \\
y + 4 &= 0 \\
\therefore y &= -4
\end{aligned}$$

The circle cuts the y-axis at $(0; -4)$.

Worked example 8: Equation of a circle with centre at $(a; b)$

QUESTION

Determine the coordinates of the centre of the circle and the length of the radius for $3x^2 + 6x + 3y^2 - 12y - 33 = 0$.

SOLUTION

Step 1: Make the coefficient of the x^2 term and the y^2 term equal to 1

The coefficient of the x^2 and y^2 term must be 1, so we take out 3 as a common factor:

$$x^2 + 2x + y^2 - 4y - 11 = 0$$

Step 2: Complete the square

Take half the **coefficient of the x term**, square it; then add and subtract it from the equation.

The coefficient of the x term is 2, so then $\left(\frac{2}{2}\right)^2 = (1)^2 = 1$.

Take half the **coefficient of the y term**, square it; then add and subtract it from the equation.

The coefficient of the y term is -4, so then $\left(\frac{-4}{2}\right)^2 = (-2)^2 = 4$.

$$\begin{aligned} x^2 + 2x + y^2 - 4y - 11 &= 0 \\ (x^2 + 2x + 1) - 1 + (y^2 - 4y + 4) - 4 - 11 &= 0 \\ (x+1)^2 + (y-2)^2 - 16 &= 0 \\ (x+1)^2 + (y-2)^2 &= 16 \end{aligned}$$

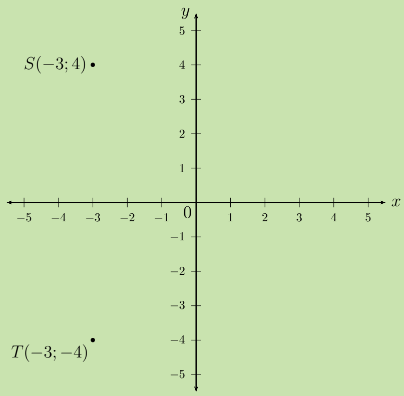
The centre of the circle is $(-1; 2)$ and the radius is 4 units.

See video: 293G at www.everythingmaths.co.za

Worked example 9: Equation of a circle with centre at $(a; b)$

QUESTION

Given $S(-3; 4)$ and $T(-3; -4)$ on the Cartesian plane.

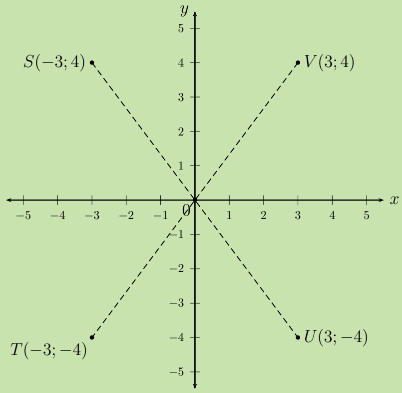


1. The points U and V are symmetrical about the origin to S and T respectively. Determine the coordinates of U and V.
2. Determine the mid-point of SU.
3. Write down the equation of the circle $STUV$.
4. Is $S\hat{T}U = 90°$? Give reasons.
5. Determine the equation of the line perpendicular to SU and passing through point S.

SOLUTION

Step 1: Determine the coordinates of U and V

For symmetry about the origin, every point $(x; y)$ is symmetrical to $(-x; -y)$. So $S(-3; 4)$ is symmetrical to $U(3; -4)$ and $T(-3; -4)$ is symmetrical to $V(3; 4)$.



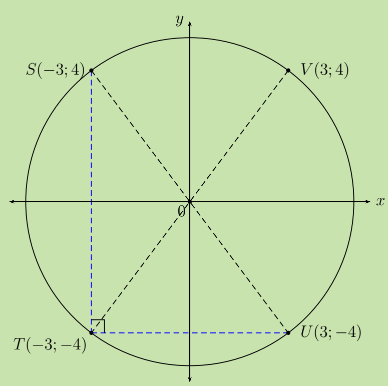
Step 2: Determine the mid-point of SU

$$\begin{aligned} M(x;y) &= \left(\frac{x_U + x_S}{2}; \frac{y_U + y_S}{2}\right) \\ &= \left(\frac{3-3}{2}; \frac{-4+4}{2}\right) \\ &= (0;0) \end{aligned}$$

The mid-point of the line SU is the origin.

Step 3: Determine the equation of the circle

$$\begin{aligned} x^2 + y^2 &= r^2 \\ (-3)^2 + (4)^2 &= r^2 \\ 9 + 16 &= r^2 \\ 25 &= r^2 \\ \therefore x^2 + y^2 &= 25 \end{aligned}$$



SU passes through the centre of the circle and is therefore a diameter. From Euclidean geometry, we know that the diameter of the circle subtends a right-angle at the circumference, therefore $S\hat{T}U = 90°$ (angle in semi-circle).

Step 4: Determine the equation of the line perpendicular to SU at point S

Determine the gradient of SU:

$$\begin{aligned} m_{SU} &= \frac{y_S - y_U}{x_S - x_U} \\ &= \frac{4 - (-4)}{-3 - 3} \\ &= \frac{8}{-6} \\ &= -\frac{4}{3} \end{aligned}$$

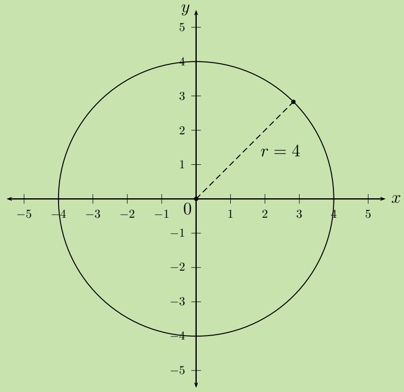
Let the gradient of the line perpendicular to SU be m_P:

$$\begin{aligned} m_{SU} \times m_P &= -1 \\ -\frac{4}{3} \times m_P &= -1 \\ \therefore m_P &= \frac{3}{4} \\ y - y_1 &= m(x - x_1) \\ \text{Substitute } S(-3;4): \quad y - 4 &= \frac{3}{4}(x - (-3)) \\ y - 4 &= \frac{3}{4}(x + 3) \\ y &= \frac{3}{4}x + \frac{25}{4} \end{aligned}$$

Worked example 10: Equation of a circle with centre at $(a;b)$

QUESTION

Given a circle with centre $(0;0)$ and a radius of 4 units.



1. If the circle is shifted 2 units down and 1 unit to the right, write down the equation of the shifted circle.
2. Sketch the original circle and the shifted circle on the same system of axes.

3. The shifted circle is reflected about the line $y = x$. Sketch the reflected circle on the same system of axes as the question above.
4. Write down the equation of the reflected circle.

SOLUTION

Step 1: Write down the equation of the circle

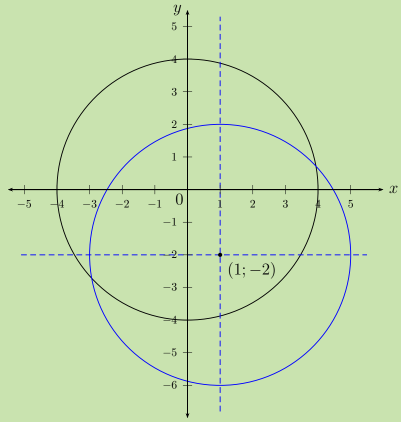
$$x^2 + y^2 = 16$$

Step 2: Determine the equation of the shifted circle

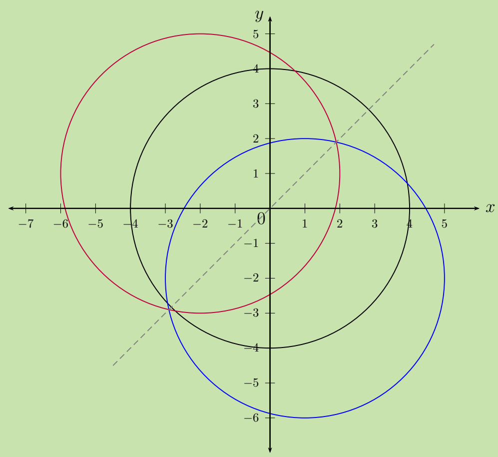
- Vertical shift: 2 units down: y is replaced with $y + 2$
- Horizontal shift: 1 unit to the right: x is replaced with $x - 1$

Therefore the equation of the shifted circle is $(x - 1)^2 + (y + 2)^2 = 16$ with centre at $(1; -2)$ and radius of 4 units.

Step 3: Draw a sketch of the circles



The shifted circle is reflected about the line $y = x$. The x and y variables are interchanged to give the circle with equation $(y-1)^2+(x+2)^2 = 16$ and centre at $(-2; 1)$.



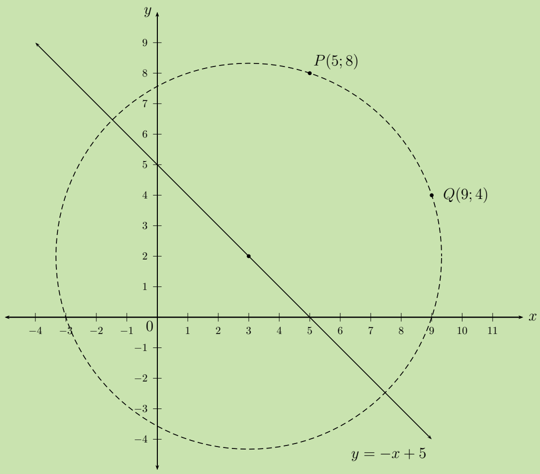
Worked example 11: Equation of a circle with centre at $(a;b)$

QUESTION

A circle with centre on the line $y = -x + 5$ passes through the points $P(5;8)$ and $Q(9;4)$. Determine the equation of the circle.

SOLUTION

Step 1: Draw a rough sketch



Step 2: Write down the general equation of a circle

$$(x - a)^2 + (y - b)^2 = r^2$$

Consider the line $y = -x+5$. Any point on this line will have the coordinates $(x; -x+5)$. Since the centre of the circle lies on the line $y = -x+5$, we can write the equation of the circle as

$$\begin{aligned}(x - a)^2 + (y - (-a + 5))^2 &= r^2 \\ (x - a)^2 + (y + a - 5)^2 &= r^2\end{aligned}$$

Step 3: Solve for the unknown variables a and r

We need two equations to solve for the two unknown variables. We substitute the two given points, $P(5;8)$ and $Q(9;4)$ and solve for a and r simultaneously:

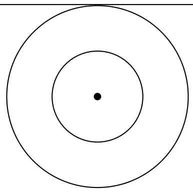
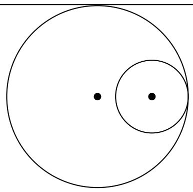
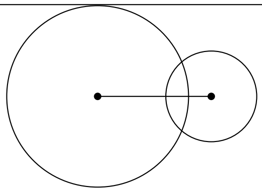
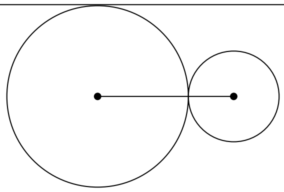
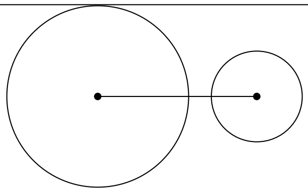
$$\begin{aligned}\text{Substitute } P(5;8): \quad (5 - a)^2 + (8 + a - 5)^2 &= r^2 \\ (5 - a)^2 + (a + 3)^2 &= r^2 \\ 25 - 10a + a^2 + a^2 + 6a + 9 &= r^2 \\ 2a^2 - 4a + 34 &= r^2 \ldots (1)\end{aligned}$$

$$\begin{aligned}\text{Substitute } Q(9;4): \quad (9 - a)^2 + (4 + a - 5)^2 &= r^2 \\ (9 - a)^2 + (a - 1)^2 &= r^2 \\ 81 - 18a + a^2 + a^2 - 2a + 1 &= r^2 \\ 2a^2 - 20a + 82 &= r^2 \ldots (2)\end{aligned}$$

$$\begin{aligned}
(1) - (2): \quad 16a - 48 &= 0 \\
16a &= 48 \\
\therefore a &= 3 \\
\text{Substitute into } (2): \quad r^2 &= 2(3)^2 - 20(3) + 82 \\
&= 18 - 60 + 82 \\
&= 40 \\
\text{And} \quad b &= -a + 5 \\
&= -3 + 5 \\
&= 2
\end{aligned}$$

Step 4: Write the final answer

The equation of the circle is $(x - 3)^2 + (y - 2)^2 = 40$.

	Circles with the same centre.
	Circles touch internally at one point: the distance between the centres is equal to the difference between the radii. $d = r_2 - r_1$
	Circles intersect at two points: the distance between the centres is less than the sum of the radii. $d < r_1 + r_2$
	Circles touch externally at one point: the distance between the centres is equal to the sum of the radii. $d = r_1 + r_2$
	Circles do not touch or intersect: the distance between the centres is greater than the sum of the radii. $d > r_1 + r_2$

Exercise 7 – 4: Equation of a circle with centre at $(a; b)$

1. Determine whether or not each of the following equations represents a circle. If not, give a reason.

   a) $x^2 + y^2 + 6y - 10 = 0$

   b) $3x^2 - 35 + 3y^2 = 9y$

   c) $40 = x^2 + 2x + 4y^2$

   d) $x^2 - 4x = \sqrt{21} + 5y + y^2$

   e) $3\sqrt{7} - x^2 - y^2 + 6y - 8x = 0$

   f) $(x-1)^2 + (y+2)^2 + 9 = 0$

2. Write down the equation of the circle:

   a) with centre $(0; 4)$ and a radius of 3 units.
   b) such that $r = 5$ and the centre is the origin.
   c) with centre $(-2; 3)$ and passing through the point $(4; 5)$.
   d) with centre $(p; -q)$ and $r = \sqrt{6}$.
   e) with $r = \sqrt{10}$ and centre $\left(-\frac{1}{2}; \frac{3}{2}\right)$ and
   f) with centre $(1; -5)$ and passing through the origin.

3. Determine the centre and the length of the radius for the following circles:

   a) $x^2 = 21 - y^2 + 4y$

   b) $y^2 + x + x^2 - \frac{15}{4} = 0$

   c) $x^2 - 4x + y^2 + 2y - 5 = 0$

   d) $x^2 + y^2 - 6y + 2x - 15 = 0$

   e) $5 - x^2 - 6x - 8y - y^2 = 0$

   f) $x^2 - \frac{2}{3}x + y^2 - 4y = \frac{35}{9}$

   g) $16x + 2y^2 - 20y + 2x^2 + 42 = 0$

   h) $6x - 6y - x^2 - y^2 = 6$

4. A circle cuts the x-axis at $R(-2; 0)$ and $S(2; 0)$. If $r = \sqrt{20}$ units, determine the possible equation(s) of the circle. Draw a sketch.
5. $P(1; 2)$ and $Q(-5; -6)$ are points on a circle such that PQ is a diameter. Determine the equation of the circle.
6. A circle with centre $N(4; 4)$ passes through the points $K(1; 6)$ and $L(6; 7)$.

   a) Determine the equation of the circle.
   b) Determine the coordinates of M, the mid-point of KL.
   c) Show that $MN \perp KL$.
   d) If P is a point on the circle such that LP is a diameter, determine the coordinates of P.
   e) Determine the equation of the line LP.

7. A circle passes through the point $A(7; -4)$ and $B(-5; -2)$. If its centre lies on the line $y + 5 = 2x$, determine the equation of the circle.
8. A circle with centre $(0; 0)$ passes through the point $T(3; 5)$.

   a) Determine the equation of the circle.
   b) If the circle is shifted 2 units to the right and 3 units down, determine the new equation of the circle.
   c) Draw a sketch of the original circle and the shifted circle on the same system of axes.
   d) On the same system of axes as the previous question, draw a sketch of the shifted circle reflected about the x-axis. State the coordinates of the centre of this circle.

9. Determine whether the circle $x^2 - 4x + y^2 - 6y + 9 = 0$ cuts, touches or does not intersect the x-axis and the y-axis.
10. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 295Q	1b. 295R	1c. 295S	1d. 295T	1e. 295V	1f. 295W
2a. 295X	2b. 295Y	2c. 295Z	2d. 2962	2e. 2963	2f. 2964
3a. 2965	3b. 2966	3c. 2967	3d. 2968	3e. 2969	3f. 296B
3g. 296C	3h. 296D	4. 296F	5. 296G	6. 296H	7. 296J
8. 296K	9. 296M				

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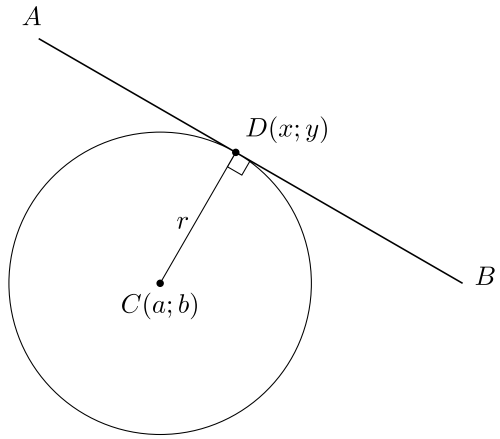
7.3 Equation of a tangent to a circle EMCHW

Investigation:

1. On a suitable system of axes, draw the circle $x^2 + y^2 = 20$ with centre at $O(0;0)$.
2. Plot the point $T(2;4)$.
3. Plot the point $P(0;5)$. Draw PT and extend the line so that is cuts the positive x-axis.
4. Measure $O\hat{T}P$.
5. Determine the gradient of the radius OT.
6. Determine the gradient of PT.
7. Prove that $PT \perp OT$.
8. Plot the point $S(2;-4)$ and join OS.
9. Draw a tangent to the circle at S.
10. Measure the angle between OS and the tangent line at S.
11. Make a conjecture about the angle between the radius and the tangent to a circle at a point on the circle.
12. Complete the sentence: the product of the . . . . . . of the radius and the gradient of the . . . . . . is equal to . . . .



A circle with centre $C(a; b)$ and a radius of r units is shown in the diagram above. $D(x; y)$ is a point on the circumference and the equation of the circle is:

$$(x - a)^2 + (y - b)^2 = r^2$$

A tangent is a straight line that touches the circumference of a circle at only one place.

The tangent line AB touches the circle at D.

The radius of the circle CD is perpendicular to the tangent AB at the point of contact D.

$$CD \perp AB$$
$$\text{and } C\hat{D}A = C\hat{D}B = 90^\circ$$

The product of the gradient of the radius and the gradient of the tangent line is equal to -1.

$$m_{CD} \times m_{AB} = -1$$

How to determine the equation of a tangent:

1. Determine the equation of the circle and write it in the form
$$(x - a)^2 + (y - b)^2 = r^2$$
2. From the equation, determine the coordinates of the centre of the circle $(a; b)$.
3. Determine the gradient of the radius:
$$m_{CD} = \frac{y_2 - y_1}{x_2 - x_1}$$
4. The radius is perpendicular to the tangent of the circle at a point D so:
$$m_{AB} = -\frac{1}{m_{CD}}$$
5. Write down the gradient-point form of a straight line equation and substitute m_{AB} and the coordinates of D. Make y the subject of the equation.
$$y - y_1 = m(x - x_1)$$

Worked example 12: Equation of a tangent to a circle

QUESTION

Determine the equation of the tangent to the circle $x^2 + y^2 - 2y + 6x - 7 = 0$ at the point $F(-2; 5)$.

SOLUTION

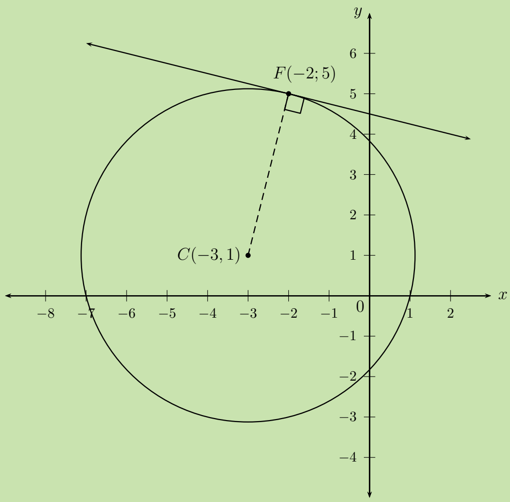
Step 1: Write the equation of the circle in the form $(x-a)^2 + (y-b)^2 = r^2$

Use the method of completing the square:

$$\begin{aligned} x^2 + y^2 - 2y + 6x - 7 &= 0 \\ x^2 + 6x + y^2 - 2y &= 7 \\ (x^2 + 6x + 9) - 9 + (y^2 - 2y + 1) - 1 &= 7 \\ (x+3)^2 + (y-1)^2 &= 17 \end{aligned}$$

Step 2: Draw a sketch

The centre of the circle is $(-3; 1)$ and the radius is $\sqrt{17}$ units.



Step 3: Determine the gradient of the radius CF

$$\begin{aligned} m_{CF} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{5-1}{-2+3} \\ &= 4 \end{aligned}$$

Step 4: Determine the gradient of the tangent

Let the gradient of the tangent line be m.

$$\begin{aligned} m_{CF} \times m &= -1 \\ 4 \times m &= -1 \\ \therefore m &= -\frac{1}{4} \end{aligned}$$

Step 5: Determine the equation of the tangent to the circle

Write down the gradient-point form of a straight line equation and substitute $m = -\frac{1}{4}$ and $F(-2;5)$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - y_1 &= -\frac{1}{4}(x - x_1) \\ \text{Substitute } F(-2;5): \quad y - 5 &= -\frac{1}{4}(x - (-2)) \\ y - 5 &= -\frac{1}{4}(x + 2) \\ y &= -\frac{1}{4}x - \frac{1}{2} + 5 \\ &= -\frac{1}{4}x + \frac{9}{2} \end{aligned}$$

Step 6: Write the final answer

The equation of the tangent to the circle at F is $y = -\frac{1}{4}x + \frac{9}{2}$.

Worked example 13: Equation of a tangent to a circle

QUESTION

The straight line $y = x + 4$ cuts the circle $x^2 + y^2 = 26$ at P and Q.

1. Calculate the coordinates of P and Q.
2. Sketch the circle and the straight line on the same system of axes. Label points P and Q.
3. Determine the coordinates of H, the mid-point of chord PQ.
4. If O is the centre of the circle, show that $PQ \perp OH$.
5. Determine the equations of the tangents to the circle at P and Q.
6. Determine the coordinates of S, the point where the two tangents intersect.
7. Show that S, H and O are on a straight line.

SOLUTION

Step 1: Determine the coordinates of P and Q

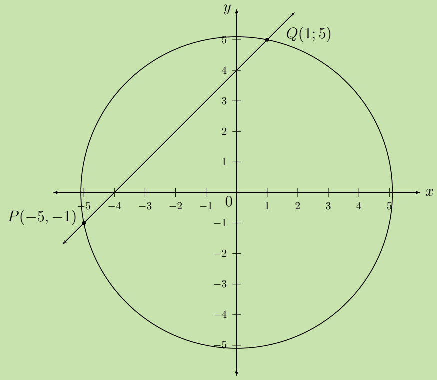
Substitute the straight line $y = x + 4$ into the equation of the circle and solve for x:

$$\begin{aligned} x^2 + y^2 &= 26 \\ x^2 + (x+4)^2 &= 26 \\ x^2 + x^2 + 8x + 16 &= 26 \\ 2x^2 + 8x - 10 &= 0 \\ x^2 + 4x - 5 &= 0 \\ (x-1)(x+5) &= 0 \\ \therefore x &= 1 \text{ or } x = -5 \end{aligned}$$

$$\begin{aligned} \text{If } x = 1 &\quad y = 1 + 4 = 5 \\ \text{If } x = -5 &\quad y = -5 + 4 = -1 \end{aligned}$$

This gives the points $P(-5;-1)$ and $Q(1;5)$.

Step 2: Draw a sketch



Step 3: Determine the coordinates of the mid-point H

$$\begin{aligned} H(x;y) &= \left(\frac{x_1+x_2}{2}; \frac{y_1+y_2}{2}\right) \\ &= \left(\frac{1-5}{2}; \frac{5-1}{2}\right) \\ &= \left(\frac{-4}{2}; \frac{4}{2}\right) \\ &= (-2;2) \end{aligned}$$

Step 4: Show that OH is perpendicular to PQ

We need to show that the product of the two gradients is equal to -1. From the given equation of PQ, we know that $m_{PQ} = 1$.

$$\begin{aligned} m_{OH} &= \frac{2-0}{-2-0} \\ &= -1 \\ m_{PQ} \times m_{OH} &= -1 \\ \therefore PQ &\perp OH \end{aligned}$$

Step 5: Determine the equations of the tangents at P and Q

Tangent at P:

Determine the gradient of the radius OP:

$$\begin{aligned} m_{OP} &= \frac{-1-0}{-5-0} \\ &= \frac{1}{5} \end{aligned}$$

The tangent of a circle is perpendicular to the radius, therefore we can write:

$$\begin{aligned} \frac{1}{5} \times m_P &= -1 \\ \therefore m_P &= -5 \end{aligned}$$

Substitute $m_P = -5$ and $P(-5;-1)$ into the equation of a straight line.

$$y - y_1 = -5(x - x_1)$$

Substitute $P(-5;-1)$:

$$\begin{aligned} y + 1 &= -5(x+5) \\ y &= -5x - 25 - 1 \\ &= -5x - 26 \end{aligned}$$

Tangent at Q:

Determine the gradient of the radius OQ:

$$\begin{aligned} m_{OQ} &= \frac{5-0}{1-0} \\ &= 5 \end{aligned}$$

The tangent of a circle is perpendicular to the radius, therefore we can write:

$$\begin{aligned} 5 \times m_Q &= -1 \\ \therefore m_Q &= -\frac{1}{5} \end{aligned}$$

Substitute $m_Q = -\frac{1}{5}$ and $Q(1;5)$ into the equation of a straight line.

$$y - y_1 = -\frac{1}{5}(x - x_1)$$

Substitute $Q(1;5)$:

$$\begin{aligned} y - 5 &= -\frac{1}{5}(x-1) \\ y &= -\frac{1}{5}x + \frac{1}{5} + 5 \\ &= -\frac{1}{5}x + \frac{26}{5} \end{aligned}$$

The equations of the tangents are $y = -5x - 26$ and $y = -\frac{1}{5}x + \frac{26}{5}$.

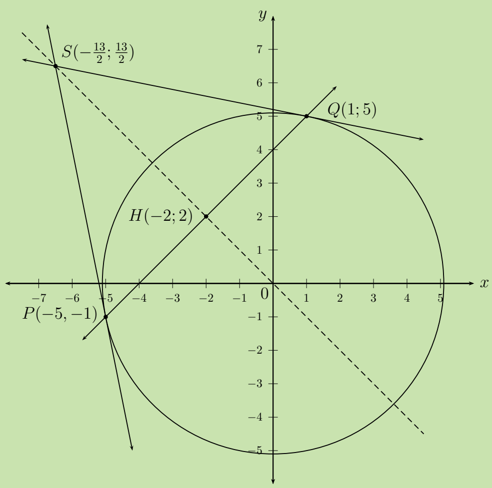
Step 6: Determine the coordinates of S

Equate the two linear equations and solve for x:

$$\begin{aligned} -5x - 26 &= -\frac{1}{5}x + \frac{26}{5} \\ -25x - 130 &= -x + 26 \\ -24x &= 156 \\ x &= -\frac{156}{24} \\ &= -\frac{13}{2} \end{aligned}$$

$$\text{If } x = -\frac{13}{2} \quad \begin{aligned} y &= -5\left(-\frac{13}{2}\right) - 26 \\ &= \frac{65}{2} - 26 \\ &= \frac{13}{2} \end{aligned}$$

This gives the point $S\left(-\frac{13}{2}; \frac{13}{2}\right)$.



Step 7: Show that S, H and O are on a straight line

We need to show that there is a constant gradient between any two of the three points. We have already shown that PQ is perpendicular to OH, so we expect the gradient of the line through S, H and O to be -1.

$$\begin{aligned} m_{SH} &= \frac{\frac{13}{2} - 2}{-\frac{13}{2} + 2} \\ &= -1 \end{aligned}$$

$$\begin{aligned} m_{SO} &= \frac{\frac{13}{2} - 0}{-\frac{13}{2} - 0} \\ &= -1 \end{aligned}$$

Therefore S, H and O all lie on the line $y = -x$.

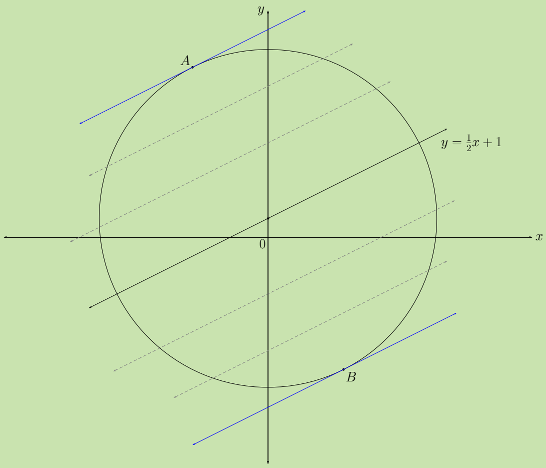
Worked example 14: Equation of a tangent to a circle

QUESTION

Determine the equations of the tangents to the circle $x^2 + (y-1)^2 = 80$, given that both are parallel to the line $y = \frac{1}{2}x + 1$.

SOLUTION

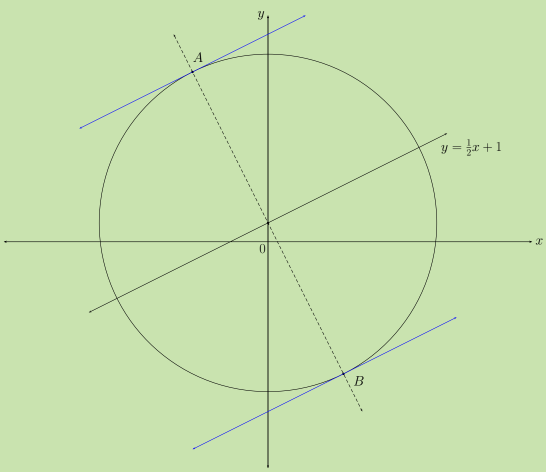
Step 1: Draw a sketch



The tangents to the circle, parallel to the line $y = \frac{1}{2}x + 1$, must have a gradient of $\frac{1}{2}$. From the sketch we see that there are two possible tangents.

Step 2: Determine the coordinates of A and B

To determine the coordinates of A and B, we must find the equation of the line perpendicular to $y = \frac{1}{2}x + 1$ and passing through the centre of the circle. This perpendicular line will cut the circle at A and B.



$$y = \frac{1}{2}x + 1$$
$$\therefore m = \frac{1}{2}$$
$$m_{\perp} = -\frac{1}{m} = -2$$
$$\therefore y = -2x + 1$$

Notice that the line passes through the centre of the circle.

To determine the coordinates of A and B, we substitute the straight line $y = -2x + 1$ into the equation of the circle and solve for x:

$$x^2 + (y-1)^2 = 80$$
$$x^2 + (-2x + 1 - 1)^2 = 80$$
$$x^2 + 4x^2 = 80$$
$$5x^2 = 80$$
$$x^2 = 16$$
$$\therefore x = \pm 4$$
$$\text{If } x = 4 \quad y = -2(4) + 1 = -7$$
$$\text{If } x = -4 \quad y = -2(-4) + 1 = 9$$

This gives the points $A(-4; 9)$ and $B(4; -7)$.

Step 3: Determine the equations of the tangents to the circle
Tangent at A:

$$y - y_1 = \frac{1}{2}(x - x_1)$$
$$y - 9 = \frac{1}{2}(x + 4)$$
$$y = \frac{1}{2}x + 11$$

Tangent at B:

$$y - y_1 = \frac{1}{2}(x - x_1)$$
$$y + 7 = \frac{1}{2}(x - 4)$$
$$y = \frac{1}{2}x - 9$$

The equation of the tangent at point A is $y = \frac{1}{2}x + 11$ and the equation of the tangent at point B is $y = \frac{1}{2}x - 9$.

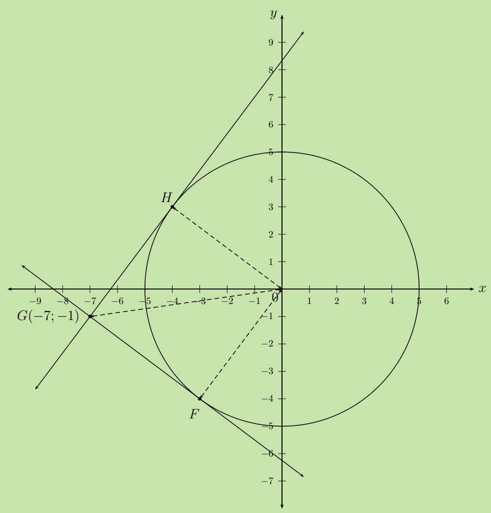
Worked example 15: Equation of a tangent to a circle

QUESTION

Determine the equations of the tangents to the circle $x^2 + y^2 = 25$, from the point $G(-7;-1)$ outside the circle.

SOLUTION

Step 1: Draw a sketch



Step 2: Consider where the two tangents will touch the circle

Let the two tangents from G touch the circle at F and H.

$$\begin{aligned} OF = OH &= 5 \text{ units} \quad \text{(equal radii)} \\ OG &= \sqrt{(0+7)^2 + (0+1)^2} \\ &= \sqrt{50} \\ GF &= \sqrt{(x+7)^2 + (y+1)^2} \\ \therefore GF^2 &= (x+7)^2 + (y+1)^2 \\ \text{And } G\hat{F}O = G\hat{H}O &= 90^\circ \end{aligned}$$

Consider $\triangle GFO$ and apply the theorem of Pythagoras:

$$\begin{aligned} GF^2 + OF^2 &= OG^2 \\ (x+7)^2 + (y+1)^2 + 5^2 &= \left(\sqrt{50}\right)^2 \\ x^2 + 14x + 49 + y^2 + 2y + 1 + 25 &= 50 \\ x^2 + 14x + y^2 + 2y + 25 &= 0 \ldots\ldots (1) \\ \text{Substitute } y^2 &= 25 - x^2 \text{ into equation } (1) \\ x^2 + 14x + \left(25 - x^2\right) + 2\left(\sqrt{25 - x^2}\right) + 25 &= 0 \\ 14x + 50 &= -2\left(\sqrt{25 - x^2}\right) \\ 7x + 25 &= -\sqrt{25 - x^2} \end{aligned}$$

$$\text{Square both sides: } (7x+25)^2 = \left(-\sqrt{25-x^2}\right)^2$$
$$49x^2 + 350x + 625 = 25 - x^2$$
$$50x^2 + 350x + 600 = 0$$
$$x^2 + 7x + 12 = 0$$
$$(x+3)(x+4) = 0$$
$$\therefore x = -3 \text{ or } x = -4$$
$$\text{At } F: x = -3 \quad y = -\sqrt{25-(-3)^2} = -\sqrt{16} = -4$$
$$\text{At } H: x = -4 \quad y = \sqrt{25-(-4)^2} = \sqrt{9} = 3$$

Note: from the sketch we see that F must have a negative y-coordinate, therefore we take the negative of the square root. Similarly, H must have a positive y-coordinate, therefore we take the positive of the square root.

This gives the points $F(-3;-4)$ and $H(-4;3)$.

Tangent at F:

$$m_{FG} = \frac{-1+4}{-7+3}$$
$$= -\frac{3}{4}$$

$$y - y_1 = m(x - x_1)$$
$$y - y_1 = -\frac{3}{4}(x - x_1)$$
$$y + 1 = -\frac{3}{4}(x + 7)$$
$$y = -\frac{3}{4}x - \frac{21}{4} - 1$$
$$y = -\frac{3}{4}x - \frac{25}{4}$$

Tangent at H:

$$m_{HG} = \frac{-1-3}{-7+4}$$
$$= \frac{4}{3}$$

$$y + 1 = \frac{4}{3}(x + 7)$$
$$y = \frac{4}{3}x + \frac{28}{3} - 1$$
$$y = \frac{4}{3}x + \frac{25}{3}$$

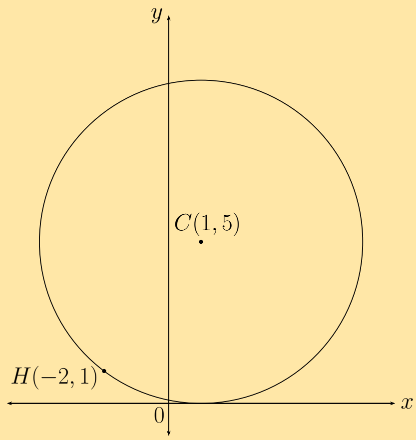
Step 3: Write the final answer

The equations of the tangents to the circle are $y = -\frac{3}{4}x - \frac{25}{4}$ and $y = \frac{4}{3}x + \frac{25}{3}$.

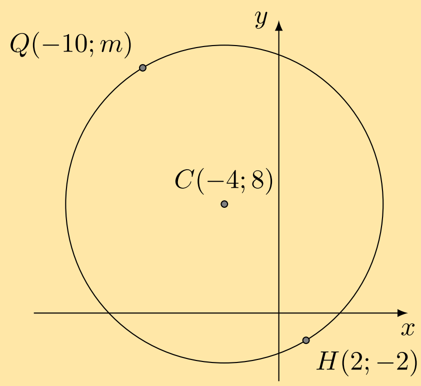
See video: 293H at www.everythingmaths.co.za

Exercise 7 – 5: Equation of a tangent to a circle

1. a) A circle with centre $(8;-7)$ and the point $(5;-5)$ on the circle are given. Determine the gradient of the radius.
   b) Determine the gradient of the tangent to the circle at the point $(5;-5)$.
2. Given the equation of the circle: $(x+4)^2 + (y+8)^2 = 136$
   a) Find the gradient of the radius at the point $(2;2)$ on the circle.
   b) Determine the gradient of the tangent to the circle at the point $(2;2)$.
3. Given a circle with the central coordinates $(a;b) = (-9;6)$. Determine the equation of the tangent to the circle at the point $(-2;5)$.
4. Given the diagram below:



   Determine the equation of the tangent to the circle with centre C at point H.
5. Given the point $P(2;-4)$ on the circle $(x-4)^2+(y+5)^2 = 5$. Find the equation of the tangent at P.
6. $C(-4;8)$ is the centre of the circle passing through $H(2;-2)$ and $Q(-10;m)$.



   a) Determine the equation of the circle.
   b) Determine the value of m.
   c) Determine the equation of the tangent to the circle at point Q.

7. The straight line $y = x + 2$ cuts the circle $x^2 + y^2 = 20$ at P and Q.
    a) Calculate the coordinates of P and Q.
    b) Determine the length of PQ.
    c) Determine the coordinates of M, the mid-point of chord PQ.
    d) If O is the centre of the circle, show that $PQ \perp OM$.
    e) Determine the equations of the tangents to the circle at P and Q.
    f) Determine the coordinates of S, the point where the two tangents intersect.
    g) Show that $PS = QS$.
    h) Determine the equations of the two tangents to the circle, both parallel to the line $y + 2x = 4$.
8. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1. 296N 2. 296P 3. 296Q 4. 296R 5. 296S 6. 296T
7. 296V

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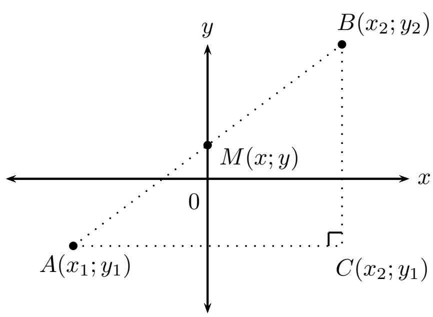
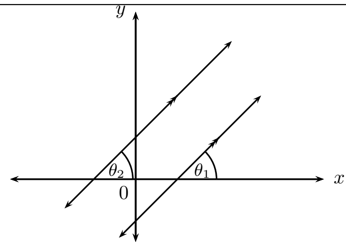
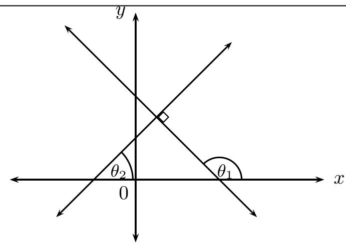
7.4 Summary EMCHX

See video: 294R at www.everythingmaths.co.za



Theorem of Pythagoras:	$AB^2 = AC^2 + BC^2$
Distance formula:	$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Gradient:	$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$ or $m_{AB} = \frac{y_1 - y_2}{x_1 - x_2}$
Mid-point of a line segment:	$M(x; y) = \left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$
Points on a straight line:	$m_{AB} = m_{AM} = m_{MB}$

Straight line equations	Formulae
Two-point form:	$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$
Gradient-point form:	$y-y_1=m(x-x_1)$
Gradient-intercept form:	$y=mx+c$
Horizontal lines:	$y=k$
Vertical lines:	$x=k$

Parallel lines		$m_1=m_2$	$\theta_1=\theta_2$
Perpendicular lines		$m_1 \times m_2=-1$	$\theta_1=90^\circ+\theta_2$

- Inclination of a straight line: the gradient of a straight line is equal to the tangent of the angle formed between the line and the positive direction of the x-axis.

$$m=\tan\theta \qquad \text{for } 0^\circ \leq \theta < 180^\circ$$

- Equation of a circle with centre at the origin:
  If $P(x;y)$ is a point on a circle with centre $O(0;0)$ and radius r, then the equation of the circle is:
  $$x^2+y^2=r^2$$

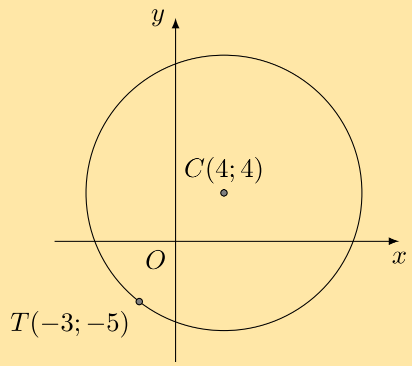
- General equation of a circle with centre at $(a;b)$:
  If $P(x;y)$ is a point on a circle with centre $C(a;b)$ and radius r, then the equation of the circle is:
  $$(x-a)^2+(y-b)^2=r^2$$

- A tangent is a straight line that touches the circumference of a circle at only one point.

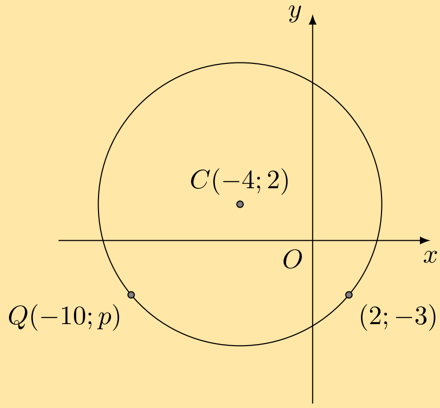
- The radius of a circle is perpendicular to the tangent at the point of contact.

Exercise 7 – 6: End of chapter exercises

1. Find the equation of the circle:
   a) with centre $(0;5)$ and radius 5
   b) with centre $(2;0)$ and radius 4
   c) with centre $(-5;7)$ and radius 18
   d) with centre $(-2;0)$ and diameter 6
   e) with centre $(-5;-3)$ and radius $\sqrt{3}$
2. a) Find the equation of the circle with centre $(2;1)$ which passes through $(4;1)$.
   b) Where does it cut the line $y = x + 1$?
3. a) Find the equation of the circle with centre $(-3;-2)$ which passes through $(1;-4)$.
   b) Find the equation of the circle with centre $(3;1)$ which passes through $(2;5)$.
4. Find the centre and radius of the following circles:
   a) $(x+9)^2 + (y-6)^2 = 36$
   b) $\frac{1}{2}(x-2)^2 + \frac{1}{2}(y-9)^2 = 1$
   c) $(x+5)^2 + (y+7)^2 = 12$
   d) $x^2 + (y+4)^2 = 23$
   e) $3(x-2)^2 + 3(y+3)^2 = 12$
5. Find the x and y intercepts of the following graphs:
   a) $x^2 + (y-6)^2 = 100$
   b) $(x+4)^2 + y^2 = 16$
6. Find the centre and radius of the following circles:
   a) $x^2 + 6x + y^2 - 12y = -20$
   b) $x^2 + 4x + y^2 - 8y = 0$
   c) $x^2 + y^2 + 8y = 7$
   d) $x^2 - 6x + y^2 = 16$
   e) $x^2 - 5x + y^2 + 3y = -\frac{3}{4}$
   f) $x^2 - 6nx + y^2 + 10ny = 9n^2$
7. a) Find the gradient of the radius between the point $(4;5)$ on the circle and its centre $(-8;4)$.
   b) Find the gradient line tangent to the circle at the point $(4;5)$.
8. a) Given $(x-1)^2 + (y-7)^2 = 10$, determine the value(s) of x if $(x;4)$ lies on the circle.
   b) Find the gradient of the tangent to the circle at the point $(2;4)$.
9. Given a circle with the central coordinates $(a;b) = (-2;-2)$. Determine the equation of the tangent line of the circle at the point $(-1;3)$.
10. Find the equation of the tangent to the circle at point T.



11. $M(-2;-5)$ is a point on the circle $x^2 + y^2 + 18y + 61 = 0$. Determine the equation of the tangent at M.
12. $C(-4;2)$ is the centre of the circle passing through $(2;-3)$ and $Q(-10;p)$.

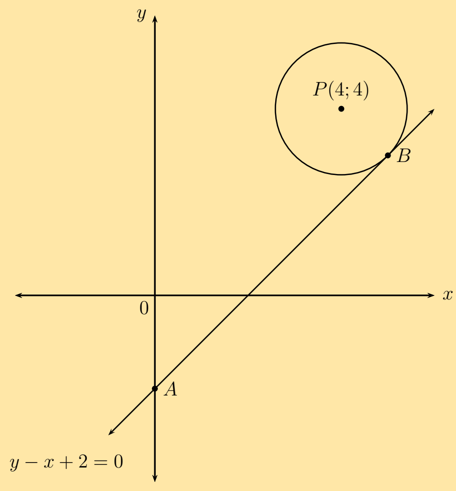


a) Find the equation of the circle given.
b) Determine the value of p.
c) Determine the equation of the tangent to the circle at point Q.

13. Find the equation of the tangent to each circle:

a) $x^2 + y^2 = 17$ at the point $(1;4)$
b) $x^2 + y^2 = 25$ at the point $(3;4)$
c) $(x+1)^2 + (y-2)^2 = 25$ at the point $(3;5)$
d) $(x-2)^2 + (y-1)^2 = 13$ at the point $(5;3)$

14. Determine the equations of the tangents to the circle $x^2 + y^2 = 50$, given that both lines have an angle of inclination of $45°$.
15. The circle with centre $P(4;4)$ has a tangent AB at point B. The equation of AB is $y - x + 2 = 0$ and A lies on the y-axis.



a) Determine the equation of PB.
b) Determine the coordinates of B.
c) Determine the equation of the circle.
d) Describe in words how the circle must be shifted so that P is at the origin.

e) If the length of PB is tripled and the circle is shifted 2 units to the right and 1 unit up, determine the equation of the new circle.

f) The equation of a circle with centre A is $x^2 + y^2 + 5 = 16x + 8y - 30$ and the equation of a circle with centre B is $5x^2 + 5y^2 = 25$. Prove that the two circles touch each other.

16. More questions. Sign in at Everything Maths online and click 'Practise Maths'.

Check answers online with the exercise code below or click on 'show me the answer'.

1a. 296W	1b. 296X	1c. 296Y	1d. 296Z	1e. 2972	2. 2973
3. 2974	4a. 2975	4b. 2976	4c. 2977	4d. 2978	4e. 2979
5a. 297B	5b. 297C	6a. 297D	6b. 297F	6c. 297G	6d. 297H
6e. 297J	6f. 297K	7. 297M	8. 297N	9. 297P	10. 297Q
11. 297R	12. 297S	13a. 297T	13b. 297V	13c. 297W	13d. 297X
14. 297Y	15. 297Z				

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