GRADE 12

Mathematics

SELF STUDY GUIDE

Trigonometry and Euclidean Geometry

basic education
Department: Basic Education
REPUBLIC OF SOUTH AFRICA

Second Chance Matric Support Programme

2030 NDP

NATIONAL EDUCATION COLLABORATION TRUST

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WOZA MATRICS 2021 CATCH UP

STAY SAFE
PROTECT SOUTH AFRICA
TOGETHER WE CAN BEAT THE CORONAVIRUS

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1. Introduction

The declaration of COVID-19 as a global pandemic by the World Health Organisation led to the disruption of effective teaching and learning in many schools in South Africa. The majority of learners in various grades spent less time in class due to the phased-in approach and rotational/ alternate attendance system that was implemented by various provinces. Consequently, the majority of schools were not able to complete all the relevant content designed for specific grades in accordance with the Curriculum and Assessment Policy Statements in most subjects.

As part of mitigating against the impact of COVID-19 on the current Grade 12, the Department of Basic Education (DBE) worked in collaboration with subject specialists from various Provincial Education Departments (PEDs) developed this Self-Study Guide. The Study Guide covers those topics, skills and concepts that are located in Grade 12, that are critical to lay the foundation for Grade 12. The main aim is to close the pre-existing content gaps in order to strengthen the mastery of subject knowledge in Grade 12. More importantly, the Study Guide will engender the attitudes in the learners to learning independently while mastering the core cross-cutting concepts.

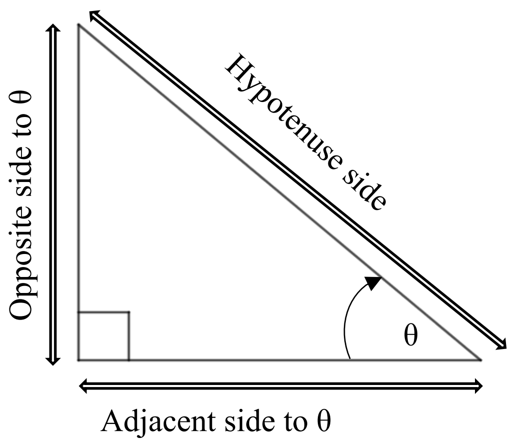
2. How to use this Self Study Guide?

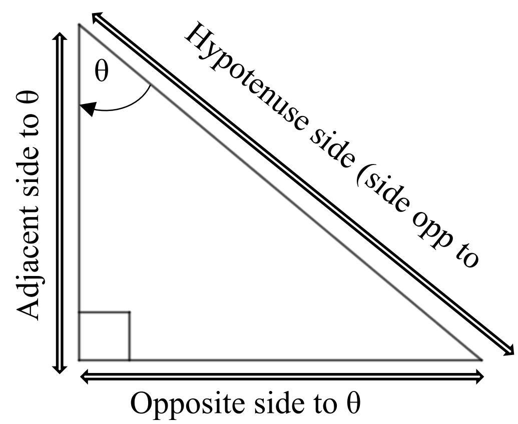
- This Self Study Guide summaries only two topics, Trigonometry and Euclidean Geometry. Hence the prescribed textbooks must be used to find more exercises.
- It highlights key concepts which must be known by all learners.
- Deeper insight into the relevance of each of the formulae and under which circumstances it can be used is very essential.
- Learners should know the variables in each formula and its role in the formula. Learners should distinguish variable in different formulae.
- More practice in each topic is very essential for you to understand mathematical concepts.
- The learners must read the question very carefully and make sure that they understand what is asked and then answer the question.
- make sure that Euclidean Geometry is covered in earlier grades. Basic work should be covered thoroughly. An explanation of the theorem must be accompanied by showing the relationship in a diagram.
- After answering all questions in this Self Study Guide, try to answer the previous question paper to gauge your understanding of the concepts your required to know.

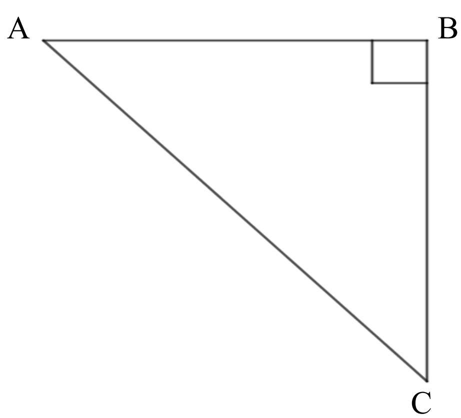
1. TRIGONOMETRY

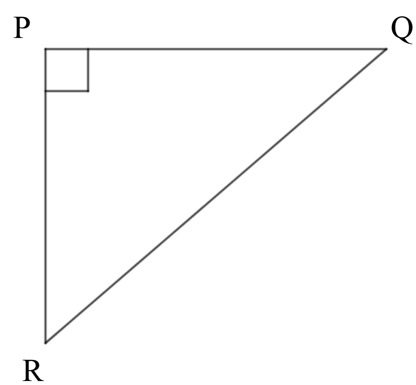
Introduction to trigonometry

Naming of sides in a right-angled triangle with respect to given angles.









In ΔABC :

1.
- AC is side opposite to 90^0 known as hypotenuse.
- AB is opposite side to $\hat{C}$
- BC is adjacent side to $\hat{C}$.

In ΔPQR :

- RQ is side opposite to 90^0 known as hypotenuse.
- PQ is opposite side to $\hat{R}$.
- PR is adjacent side to $\hat{R}$.

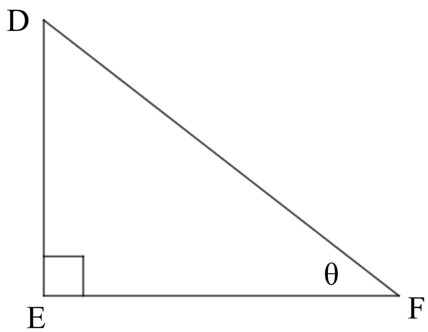
2.

- AC is side opposite to 90^0 known as hypotenuse..
- AB is adjacent side to $\hat{A}$
- BC is side oppositet to $\hat{A}$.

- RQ is side opposite to 90^0 known as hypotenuse.
- PQ is adjacent side to $\hat{Q}$.
- PR is opposite side to $\hat{Q}$

DEFINITIONS OF TRIGONOMETRIC RATIOS

Trigonometric ratios can be defined in **right-angled triangles ONLY**.



- $\sin\theta = \dfrac{opp.side\,to\,\theta}{hypotenuse} = \dfrac{DE}{DF}$
- $\cos\theta = \dfrac{adj.side\,to\,\theta}{hypotenuse} = \dfrac{EF}{DF}$
- $\tan\theta = \dfrac{opp.side\,to\,\theta}{adj.side\,to\,\theta} = \dfrac{DE}{EF}$
- $\cos ec\theta = \dfrac{hypotenuse}{opp.side\,to\,\theta} = \dfrac{DF}{DE}$
- $\sec\theta = \dfrac{hypotenuse}{adj.side\,to\,\theta} = \dfrac{DF}{EF}$
- $\cot\theta = \dfrac{adj.side\,to\,\theta}{opp.side\,to\,\theta} = \dfrac{EF}{DE}$

RECIPROCAL IDENTITIES

NB: ONLY EXAMINED IN GRADE 10

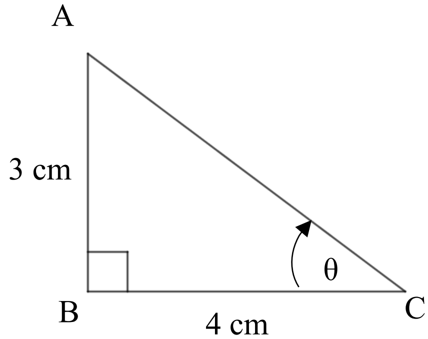
• $\sin\theta = \dfrac{1}{\cos ec\theta}$	• $\cos ec\theta = \dfrac{1}{\sin\theta}$
• $\cos\theta = \dfrac{1}{s\,ec\theta}$	• $\sec\theta = \dfrac{1}{\cos\theta}$
• $\tan\theta = \dfrac{1}{\cot\theta}$	• $\cot\theta = \dfrac{1}{\tan\theta}$

Revision grade 8, 9 and 10 work (use of Pythagoras Theorem)

Pythagoras theorem is only used in right-angled triangles: "The square on the hypotenuse is equal to the sum of the squares in the remaining two sides of a triangle".

Example 1

In the diagram below, $\triangle ABC$, $\hat{B} = 90^0, AB = 3\,cm, BC = 4cm$



1.1 Calculate the length of AC.

	Solution		
	$AC^2 = AB^2 + BC^2$ Pyth. theorem $= (3\,cm)^2 + (4\,cm)^2$ $AC = 5\,cm$	✓ correct substitution in Pyth. Theo ✓ answer	(2)

1.2	Determine the values of the following trigonometric ratios:		
	1.2.1 $\sin\theta$	$\sin\theta = \frac{AB}{AC} = \frac{3}{5}$ ✓ correct ratio	(1)
	1.2.2 $\cos\theta$	$\cos\theta = \frac{BC}{AC} = \frac{4}{5}$ ✓ correct ratio	(1)
1.3	**1.3.1** Determine the size of $\hat{A}$ in terms of θ		(1)
		$\hat{A} = 180^0 - (90^0 + \theta)$ [$\angle s\ in\ a\ \Delta$] $\hat{A} = 180^0 - 90^0 - \theta$ $\hat{A} = 90° - \theta$	
		✓ size of $\hat{A}$	(1)

1.3.2 Hence, or otherwise, determine the value of $\cos\left(90^0 - \theta\right)$ (1)

(1)

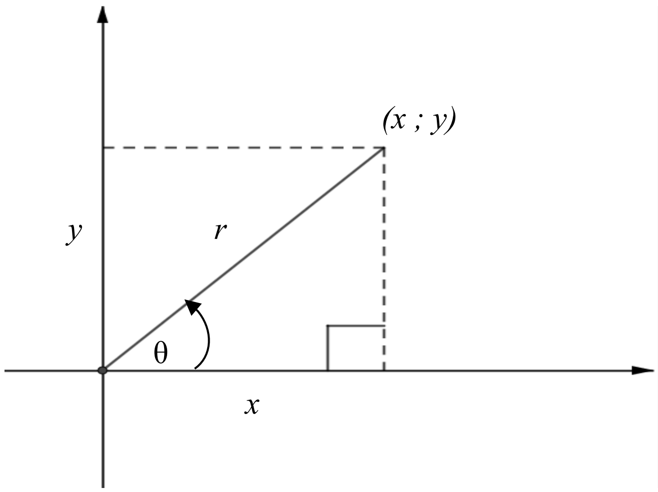
$$\cos\left(90^0 - \theta\right) = \frac{adj.side\,to\left(90^0 - \theta\right)}{hypotenuse}$$

$$= \frac{AB}{BC}$$

$$= \frac{3}{5}\ ✓$$

3.1 Revision grade 10

CARTESIAN PLANE AND IDENTITIES



NB *r* is always positive, whilst *x* and *y* can be positive or negative

Defining trig ratios in terms of *x*, *y* and *r*:

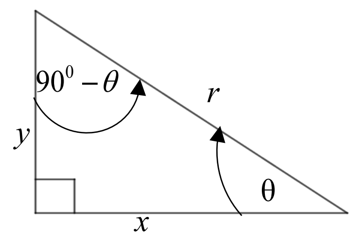
- $\sin\theta = \frac{y}{r}$ $\qquad$ $\cos ec\theta = \frac{r}{y}$
- $\cos\theta = \frac{x}{r}$ $\qquad$ $\sec\theta = \frac{r}{x}$
- $\tan\theta = \frac{y}{x}$ $\qquad$ $\cot\theta = \frac{x}{y}$

3.2 Revision grade 11

DERIVING IDENTITIES (using *x*, *y* and *r*)	
1. $\cos^2\theta+\sin^2\theta=\left(\frac{x}{r}\right)^2+\left(\frac{y}{r}\right)^2$ $=\frac{x^2+y^2}{r^2} \quad NB\left(x^2+y^2=r^2 \ Pyth\right)$ $=\frac{r^2}{r^2}$ $=1$ $\therefore \cos^2\theta+\sin^2\theta=1$	**2.** $\frac{\sin\theta}{\cos\theta}=\frac{\frac{y}{r}}{\frac{x}{r}}$ $=\frac{y}{x}$ $=\tan\theta$ $\therefore \frac{\sin\theta}{\cos\theta}=\tan\theta$

CO-RATIOS/FUNCTIONS

- $\sin\left(90^0-\theta\right)=\frac{x}{r}=\cos\theta$
- $\cos\left(90^0-\theta\right)=\frac{y}{r}=\sin\theta$



BASIC IDENTITIES

- $\cos^2\theta+\sin^2\theta=1$
- $\tan\theta=\frac{\sin\theta}{\cos\theta}$

Grade 12 Identities

COMPOUND ANGLE IDENTITIES

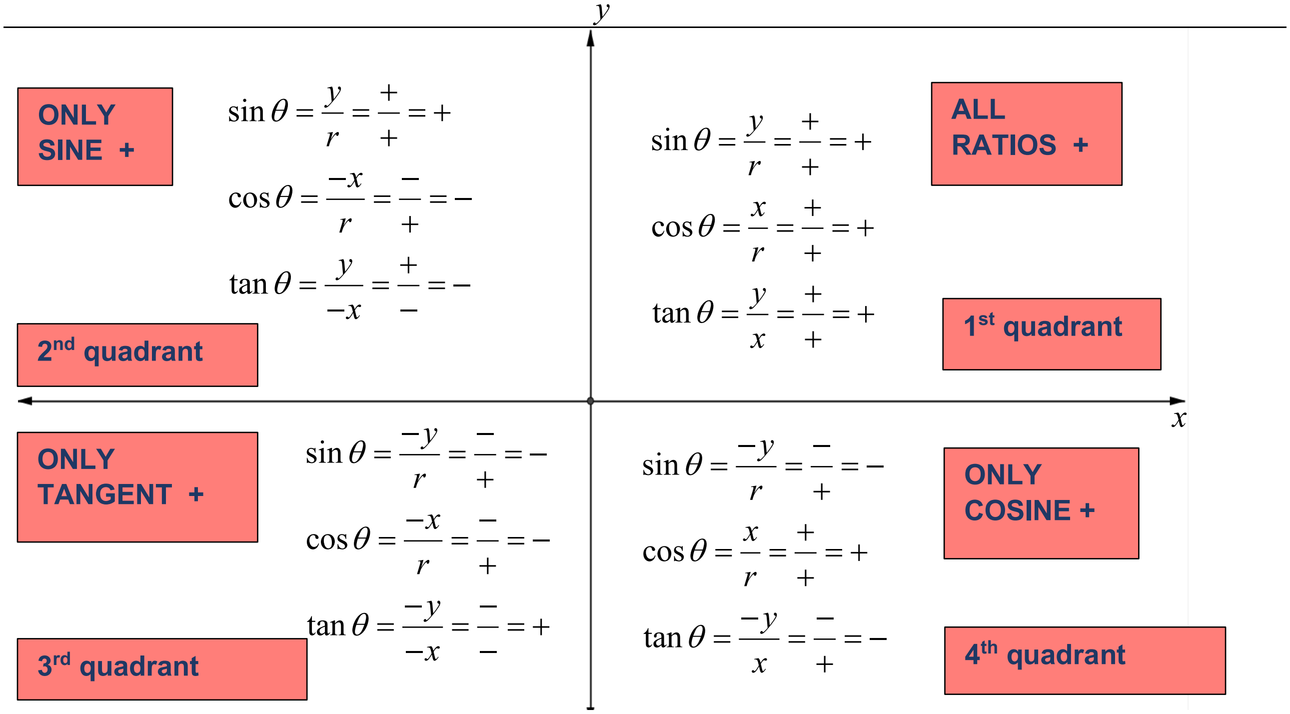
- $\cos(\alpha-\beta)=\cos\alpha.\cos\beta+\sin\alpha.\sin\beta$
- $\cos(\alpha+\beta)=\cos\alpha.\cos\beta-\sin\alpha.\sin\beta$
- $\sin(\alpha-\beta)=\sin\alpha.\cos\beta-\cos\beta.\sin\alpha$
- $\sin(\alpha+\beta)=\sin\alpha.\cos\beta+\cos\beta.\sin\alpha$

DOUBLE ANGLE IDENTITIES

- $\cos 2\alpha=\begin{cases}\cos^2\alpha-\sin^2\alpha\\1-2\sin^2\alpha\\2\cos^2\alpha-1\end{cases}$
- $\sin 2\alpha=2\sin\alpha\cos\alpha$

Proofs for the compound angle identities are examinable

SIGNS OF TRIGONOMETRIC RATIOS IN ALL THE FOUR QUADRANTS

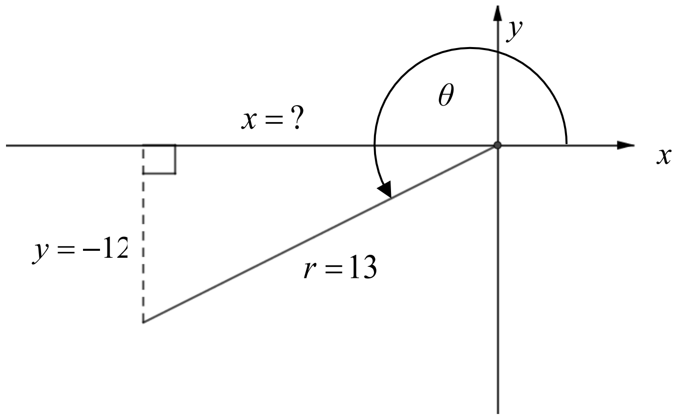


Example 1

If $\sin\theta = -\frac{12}{13}$ and $90^0 \le \theta \le 270^0$, determine the values of the following:

1. $\frac{\sin\theta}{\cos\theta}$

When answering this question, you need to define your trig ratio. Like $\sin\theta = -\frac{12}{13} = \frac{y}{r}$. Then you will know that $y = -12$ and $r = 13$, *r* will never be negative, then the negative sign will be taken by *y*. Sine is negative in 3rd and 4th quadrants. $90^0 \le \theta \le 270^0$ is an angle between 2nd and 3rd quadrants. **To know which quadrant from the two conditions, we must choose the quadrant that satisfies both conditions. Hence the 3rd quadrant.**



$r^2 = x^2 + y^2$ *Pythagoras theorem*

$x = \pm\sqrt{(13)^2 - (-12)^2}$

$= -5$ [*x in 3rd quad is neg*] ✓

$\dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{-12}{13}}{\frac{-5}{13}} = \dfrac{12}{5}$ ✓

2. $\cos^2\theta$

$\cos^2\theta = \left(\dfrac{-5}{13}\right)^2$

$= \dfrac{25}{169}$ ✓

3. $1 - \sin^2\theta$

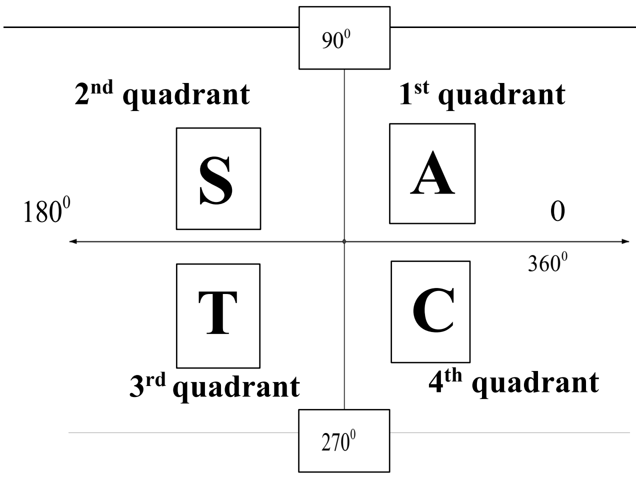
$1 - \sin^2\theta = 1 - \left(\dfrac{-12}{13}\right)^2$ ✓

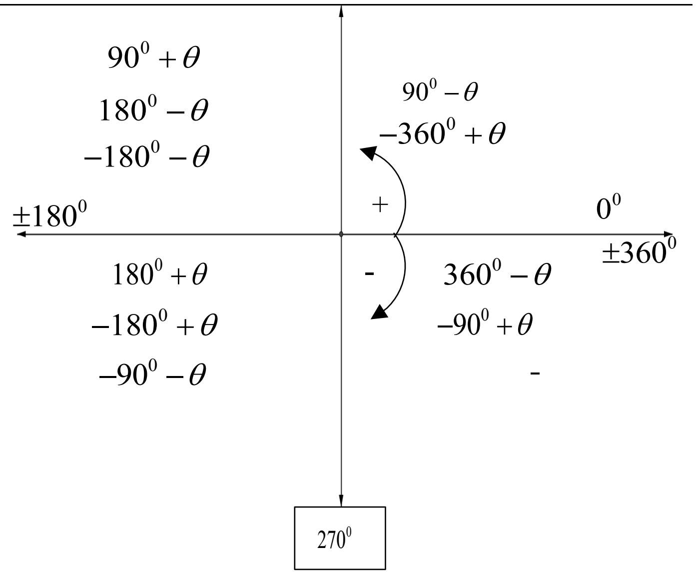
$= 1 - \dfrac{144}{169}$

$= \dfrac{25}{169}$ ✓

CAST DIAGRAM

CAST: ALL STUDENTS TAKE COFFEE

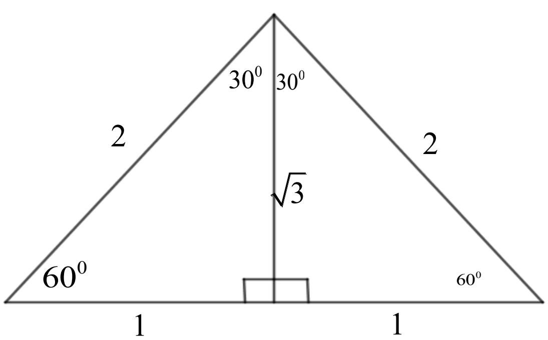


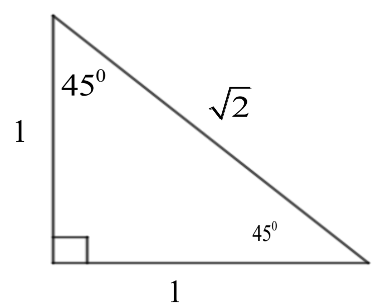


NB: We can do reduction for angles rotating clockwise by adding 360^0 up until the angle is in the range of 0^0 to 360^0.

3rd quadrant

SPECIAL ANGLES





$\sin 0^0 = 0$

$\cos 0^0 = 1$

$\tan 0^0 = 0$

$\sin 30^0 = \frac{1}{2}$

$\cos 30^0 = \frac{\sqrt{3}}{2}$

$\tan 30^0 = \frac{1}{\sqrt{3}}$

$\sin 45^0 = \frac{1}{\sqrt{2}}$

$\cos 45^0 = \frac{1}{\sqrt{2}}$

$\tan 45^0 = 1$

$\sin 60^0 = \frac{\sqrt{3}}{2}$

$\cos 60^0 = \frac{1}{2}$

$\tan 60^0 = \frac{\sqrt{3}}{1}$

$\sin 90^0 = 1$
$\cos 90^0 = 0$
$\tan 90^0 = undefined$

$\sin 180^0 = 0$
$\cos 180^0 = -1$
$\tan 180^0 = 0$

$\sin 270^0 = -1$
$\cos 270^0 = 0$
$\tan 270^0 = undefined$

$\sin 360^0 = 0$
$\cos 360^0 = 1$
$\tan 360^0 = 0$

REDUCTION FORMULAE

Identify in which quadrant the angle(s) lie first, then you will be able to know the sign of each trigonometric ratio(s) referring to CAST diagram, then change the trig ratio to its co-function if you are reducing by 90.

$90^0 - \theta$ **(1st quadrant)**	$-\theta$ **(4th quadrant)**
• $\sin(90^0 - \theta) = \cos\theta$	• $\sin(-\theta) = -\sin\theta$
• $\cos(90^0 - \theta) = \sin\theta$	• $\cos(-\theta) = \cos\theta$
• $\tan(90^0 - \theta) = \dfrac{\sin(90^0 - \theta)}{\cos(90^0 - \theta)}$ $= \dfrac{\cos\theta}{\sin\theta}$ $= \dfrac{1}{\tan\theta}$	• $\tan(-\theta) = -\tan\theta$ OR • $\tan(-\theta) = \tan(360^0 - \theta) = -\tan\theta$ Adding 360^0 **until the range is between** 0^0 and 360^0

$90^0 + \theta$ **(2nd quadrant)**	$\theta - 90^0$ **(4th quadrant)**
• $\sin(90^0 + \theta) = \cos\theta$	• $\sin(\theta - 90^0) = -\cos\theta$
• $\cos(90^0 + \theta) = -\sin\theta$	• $\cos(\theta - 90^0) = \sin\theta$

• $\tan(90^0+\theta)=-\dfrac{1}{\tan\theta}$	• $\tan(\theta-90^0)=-\dfrac{1}{\tan\theta}$

$180^0-\theta$ **(2nd quadrant)**	$-\theta-90^0$ **(3rd quadrant)**
• $\sin(180^0-\theta)=\sin\theta$	• $\sin(-\theta-90^0)=-\cos\theta$
• $\cos(180^0-\theta)=-\cos\theta$	• $\cos(-\theta-90^0)=-\sin\theta$
• $\tan(180^0-\theta)=-\tan\theta$	• $\tan(-\theta-90^0)=\dfrac{1}{\tan\theta}$

$180^0+\theta$ **(3rd quadrant)**	$\theta-180^0$ **(3rd quadrant)**
• $\sin(180^0+\theta)=-\sin\theta$	• $\sin(\theta-180^0)=-\sin\theta$ **OR** • $\sin(\theta-180^0)=\sin[-(180^0-\theta)]$ $=-\sin\theta$
• $\cos(180^0+\theta)=-\cos\theta$	
• $\tan(180^0+\theta)=\tan\theta$	• $\cos(\theta-180^0)=-\cos\theta$
	• $\tan(\theta-180^0)=\tan\theta$

$360^0-\theta$ **(4th quadrant)**	$-\theta-180^0$ **(2nd quadrant)**
• $\sin(360^0-\theta)=-\sin\theta$	• $\sin(-\theta-180^0)=\sin\theta$
• $\cos(360^0-\theta)=\cos\theta$	• $\cos(-\theta-180^0)=-\cos\theta$
• $\tan(360^0-\theta)=-\tan\theta$	• $\tan(-\theta-180^0)=-\tan\theta$

$360^0+\theta$ **(1st quadrant)**	$\theta-360^0$ **(1st quadrant)**
• $\sin(360^0+\theta)=\sin\theta$	• $\sin(\theta-360^0)=\sin\theta$

• $\cos\left(360^0+\theta\right)=\cos\theta$	• $\cos\left(\theta-360^0\right)=\cos\theta$
• $\tan\left(360^0+\theta\right)=\tan\theta$	• $\tan\left(\theta-360^0\right)=\tan\theta$

Worked-out Example 1

Write the following as ratios of θ:

Solutions

1.1. $\cos(180^0-\theta)$

1.2. $\tan(\theta-360^0)$

1.3. $\sin\left(-\theta\right)$

1.1 $\cos(180^0-\theta)=-\cos\theta$ ✓

1.2 $\tan(\theta-360^0)=\tan\theta$ ✓

1.3 $\sin\left(-\theta\right)=-\sin\theta$ ✓

Worked-out Example 2

Express the following as ratios of acute angles

Solutions

2.1 $\tan 130^0$

2.1 $\tan 130^0=\tan\left(180^0-50^0\right)$

$=-\tan 50^0$ ✓

130^0 cannot just be written in any way but in terms of 90^0 or 180^0. It is in the 2nd quadrant and is greater than 90^0 but less than 180^0 .
$\therefore 130^0=180^0-50^0$ *OR* $130^0=90^0+40^0$. In this case we choose expression by 180^0 as our ratios remain the same in 180^0. We can write $130^0=90^0+40^0$ but bear in mind that our ratios change to their co-ratios when reducing by 90^0.

2.2 $\cos\left(-284\right)^0$

2.2 $\cos\left(-284\right)^0=\cos(76-360^0)$

$=\cos 76^0$ ✓

OR

$$\cos\left(-284\right)^{0} = \cos(-284 + 360^{0})$$

$$= \cos 76^{0} \checkmark$$

Explanation in 2.1 also applies in 2.2 according the quadrant where the angle lies.

Worked-out Example 3

Simplify the following expressions:

3.1 $$\frac{\cos\left(-\theta\right)\cos(90^{0}+\theta)\tan\left(\theta+180^{0}\right)}{\tan\left(360^{0}-\theta\right).\cos\theta.\sin\left(360^{0}+\theta\right)}$$

Solutions

$$\frac{\cos\left(-\theta\right)\cos(90^{0}+\theta)\tan\left(\theta+180^{0}\right)}{\tan\left(360^{0}-\theta\right).\cos\theta.\sin\left(360^{0}+\theta\right)}$$

$$= \frac{\left(\cos\theta\right).\left(-\sin\theta\right).\left(\tan\theta\right)}{\left(-\tan\theta\right).\cos\theta.\left(\sin\theta\right)}$$

$$= 1$$

$\checkmark \cos\theta$

$\checkmark -\sin\theta$

$\checkmark \tan\theta$

$\checkmark -\tan\theta$

$\checkmark \sin\theta$

$\checkmark 1$

3.2 $$\frac{2\sin 40^{0}.\cos\left(-50^{0}\right)}{\sin 80^{0}}$$

Solutions

$\checkmark \sin 40^{0}$

$$= \frac{2\sin 40^0 . \cos\left(40^0 - 90^0\right)}{2\sin 40^0 \cos 40^0}$$

$$= \frac{2\sin 40^0 . \left(\sin 40^0\right)}{2\sin 40^0 . \cos 40^0}$$

$$= \tan 40^0$$

✓ $2\sin 40^0 \cos 40^0$

✓ $\tan 40^0$

3.3

$$\sqrt{\frac{\frac{1}{2}\sin 2x}{\tan\left(540^0 + x\right).\left[\frac{1}{\cos^2 x} - \tan^2 x\right]}}$$

✓ $2\sin x.\cos x$

$$= \sqrt{\frac{\frac{1}{2}.2\sin x.\cos x}{\left(\tan x\right).\left[\frac{1 - \cos^2 x.\frac{\sin^2 x}{\cos^2 x}}{\cos^2 x}\right]}}$$

✓ $\tan x$ ✓ $\frac{\sin^2 x}{\cos^2 x}$

$$= \sqrt{\frac{\sin x.\cos x}{\frac{\sin x}{\cos x}.\left[\frac{1 - \sin^2 x}{\cos^2 x}\right]}}$$

✓ $\frac{\sin x}{\cos x}$

✓ $1 - \sin^2 x = \cos^2 x$

$$= \sqrt{\frac{\sin x \cos x}{\frac{\sin x}{\cos x}.\frac{\cos^2 x}{\cos^2 x}}}$$

$$= \sqrt{\sin x.\cos x.\frac{\cos x}{\sin x}}$$

✓ simplification

$$= \sqrt{\cos^2 x}$$

$$= \cos x$$

✓ $\cos x$

Worked-out Example 4

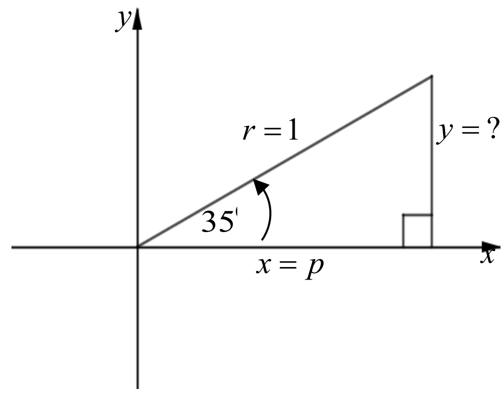
If $\cos 35^0 = p$, determine the following in terms of *p*:

Solutions

4.1 $\sin 35^0$

According to the definition of cosine, *p* represents *x* and 1 represents *r*

4.1 $\cos 35^0 = \frac{p}{1} = \frac{x}{r}$



NB: Consider finding the 3rd angle before you start doing your calculations.

$y = \sqrt{(1)^2 - (p)^2}$ *Pythagoras*

$= \sqrt{1 - p^2}$

$\therefore \sin 35^0 = \frac{y}{r}$

$= \frac{\sqrt{1-p^2}}{1}$

$= \sqrt{1-p^2}$

$\therefore \sin 35^0 = \frac{\sqrt{1-p^2}}{1}$

✓ $y = \sqrt{1-p^2}$

✓ $\sin 35^0 = \frac{\sqrt{1-p^2}}{1}$

4.2	$\tan 215^0 + \sin(-55^0)$ $= \tan(180^0 + 35^0) + (-\sin 55^0)$ $= (\tan 35^0) + (-\sin 55^0)$ $= \left(\frac{\sqrt{1-p^2}}{p}\right) + \left(-\frac{p}{1}\right)$ $= \frac{\sqrt{1-p^2} - p^2}{p}$	✓ $\tan 35^0$ ✓ $\sin 55^0$ ✓✓ substitution

HOW TO USE DOUBLE ANGLE IDENTITIES [always change double angles to single angle to make your expression to be in a more simplified form]

1. **If you see** $\sin 2x$ **always substitute it by** $2\sin\cos x$**. There is only 1 option for** $\sin 2x$**.**

2. $\cos 2x$ **has 3 options**

2.1 **If you see** $\sin x$ **before or after** $\cos 2x$**, replace** $\cos 2x$ **by** $1-2\sin^2 x$

2.2 **If you see** $\cos x$ **before or after** $\cos 2x$**, replace** $\cos 2x$ **by** $2\cos^2 x - 1$

2.3 **If you see** $\sin x \cos x$ **before or after** $\cos 2x$**, replace** $\cos 2x$ **by** $\cos^2 x - \sin^2 x$

2.4 **If you see** ± 1 **before or after** $\cos 2x$ **.try to eliminate it by replacing by** $1-2\sin^2 x$ **or** $2\cos^2 x - 1$**.**

$$\left[\begin{array}{l} \cos 2x - 1 = \left(1 - 2\sin^2 x\right) - 1 \\ \qquad\qquad = -2\sin^2 x \\ \qquad\qquad \text{OR} \\ 1 - \cos 2x = 1 - \left(2\cos^2 x - 1\right) \\ \qquad\qquad = -2\cos^2 x \\ \qquad\qquad \text{OR} \\ \cos 2x + 1 = 2\cos^2 x - 1 + 1 \\ \qquad\qquad = 2\cos^2 x \end{array}\right]$$

For example:

By doing so you will be left with 1 term. Replace $\cos 2x$ by the identity which will be the additive inverse to ± 1

TRIGONOMETRIC IDENTITIES INCLUDING DOUBLE ANGLES

Worked-out Example1: Prove that $\dfrac{\sin 2x}{\cos 2x+1}=\tan x$

$$
\begin{aligned}
LHS &= \frac{\sin 2x}{\cos 2x+1}\\
&= \frac{2\sin x\cos x}{2\cos^2 x-1+1}\\
&= \frac{2\sin x\cos x}{2\cos^2 x}\\
&= \frac{\sin x}{\cos x}\\
&= \tan x\\
&= RHS
\end{aligned}
$$

- Numerator has 1 option only
- Denominator $\cos 2x+1$, then eliminate 1 by replacing $\cos 2x$ by $2\cos^2 x-1$
- Simplify

Worked-out Example 2: Prove that $\dfrac{\sin 2x-\cos x}{\sin x-\cos 2x}=\dfrac{\cos x}{\sin x+1}$

$$
\begin{aligned}
LHS &= \frac{\sin 2x-\cos x}{\sin x-\cos 2x}\\
&= \frac{2\sin x\cos x-\cos x}{\sin x-\left(1-2\sin^2 x\right)}\\
&= \frac{\cos x\left(2\sin x-1\right)}{\sin x-1+2\sin^2 x}\\
&= \frac{\cos x\left(2\sin x-1\right)}{2\sin^2 x+\sin x-1}\\
&= \frac{\cos x\left(2\sin x-1\right)}{\left(2\sin x-1\right)\left(\sin x+1\right)}\\
&= \frac{\cos x}{\sin x+1}\\
&= RHS
\end{aligned}
$$

- $\sin 2x$ has only 1 option
- Denominator has $\sin x$, then replace $\cos 2x$ by $1-2\sin^2 x$
- Denominator must be written in standard form
- Factorise numerator and denominator

Worked-out Example 3: Prove that $\cos 4x = 8\cos^4 x - \cos^2 x + 1$

$$\begin{aligned} LHS &= \cos 4x \\ &= 2\cos^2 2x - 1 \\ &= 2\left(2\cos^2 x - 1\right)^2 - 1 \\ &= 2\left(4\cos^4 x - 4\cos^2 x + 1\right) - 1 \\ &= 8\cos^4 x - 8\cos^2 x + 2 - 1 \\ &= 8\cos^4 x - 8\cos^2 x + 1 \end{aligned}$$

Worked-out Example 1 Expressions

1.1 Determine, **without using a calculator**, the value of the following trigonometric expression:

$$\frac{\sin 2x.\cos(-x) + \cos 2x.\sin(360° - x)}{\sin(180° + x)}$$

Reduction of angles CAST diagram

Solutions

$$\frac{\sin 2x.\cos(-x) + \cos 2x.\sin(360° - x)}{\sin(180° + x)}$$

$$= \frac{\sin 2x.cox + \cos 2x.(-\sin x)}{(-\sin x)}$$

$$= \frac{\sin 2x.\cos x - \cos 2x \sin x}{-\sin x}$$

$$= \frac{\sin(2x - x)}{-\sin x}$$

$$= \frac{\sin x}{-\sin x}$$

$$= -1$$

Recognition of compound angle

Worked-out

Example 2

Solutions

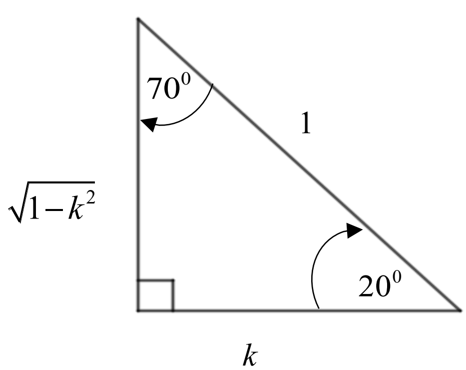
2.1 Prove that $\cos 15^0 = \dfrac{\sqrt{2}\left(\sqrt{3}+1\right)}{4}$ without using a calculator

$$\text{LHS} = \cos 15^0$$
$$= \cos\left(45^0 - 30^0\right)$$
$$= \cos 45^0 \cos 30^0 + \sin 45^0 \sin 30^0$$
$$= \frac{\sqrt{2}}{2} . \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} . \frac{1}{2}$$
$$= \frac{\sqrt{2}\left(\sqrt{3}+1\right)}{4}$$

Express 15^0 **in terms of special angles as you are told to prove without using a calculator.** 15^0 **is an acute angle that can be expressed in terms of** $15^0 = 45^0 - 30^0 \ or \ 15^0 = 60^0 - 45^0$**. From there, these angles are forming compound angles. We cannot reduce** 15^0 **as it is acute angle already. Compound angle identity for** $\cos\left(45^0 - 30^0\right)$ **needs to be applied.**

Worked-out Example 3

3.1 If $\tan 20^0 = k$, determine, **without using a calculator**, expressions in terms of k :



Solutions

Express 40^0 in terms 20^0 as they are related. $40^0 = 2 \times 20^0$, then apply double angle identities

3.1 $\sin 40^0$

$\sin 40^0 = 2\sin 20^0.\cos 20^0$

$= 2.\frac{\sqrt{1-k^2}}{1}.k$

$= 2k\sqrt{1-k^2}$

Express 35^0 in terms 70^0 as they are related. $35^0 = \frac{1}{2}\times 70^0$, then apply double angle identities

3.2 $\cos 35^0$

$\cos 2\alpha = 2\cos^2\alpha - 1$
$\alpha = 35^0$

$\cos 35^0$

$\cos 70^0 = 2\cos^2 35^0 - 1$

$\cos 70^0 + 1 = 2\cos^2 35^0$

$\frac{\cos 70^0 + 1}{2} = \cos^2 35^0$

$\cos 35^0 = \sqrt{\frac{\cos 70^0 + 1}{2}}$

$= \sqrt{\frac{\sqrt{1-k^2}+1}{2}}$

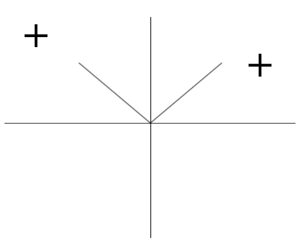
$= \sqrt{\frac{\sqrt{2-k^2}}{2}}$

TRIGONOMETRIC EQUATIONS

Solving of equations if $0^0 \leq x \leq 360^0$

Any trigonometric function is positive in two quadrants and negative in two quadrants; so there will always be two solutions if $0^0 \leq x \leq 360^0$**. The sign of the ratio tells us in which quadrant the angle is. Reference angle is an acute angle that is always positive irrespective of the ratio.**

If $\sin x = +ratio$, $0 \leq ratio \leq 1$



$1st : x = ref\angle$

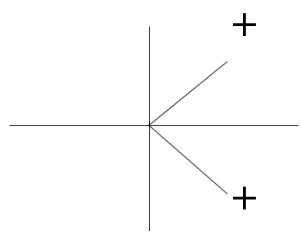
or

$2nd : x = 180^0 - ref\angle$

NB: $1st : x \neq 90^0 - ref\angle$, $2nd : x \neq 90^0 + ref\angle$

$90^0 + x$ is in the 2nd quad but **it is not advisable to** use 90 as our ratios are changing to their co-ratios

If $\cos x = +$ ratio $0 \leq ratio \leq 1$



$1st : x = ref\angle$

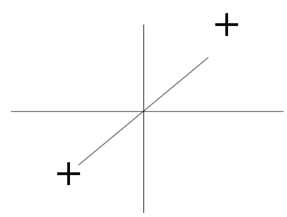
or

$4th : x = 360^0 - ref\angle$

NB: $1st : x \neq 90^0 - ref\angle$, $4th : x \neq 270^0 + ref\angle$

$270^0 + x$ is in the 4th quad but **it is not advisable to** use 270 as our ratios are changing to their co-ratios as well as 90

If $\tan x = +$ ratio, $ratio \geq 0$



$1st : x = ref\angle$

or

$3rd : x = 180^0 + ref\angle$

NB: $1st : x \neq 90^0 - ref\angle$, $3rd : x \neq 270^0 - ref\angle$

$270^0 - x$ is in the 3rd quad but **it is not advisable to** use 270 as our ratios are changing to their co-ratios as well as 90

If $\sin x = -ratio$,

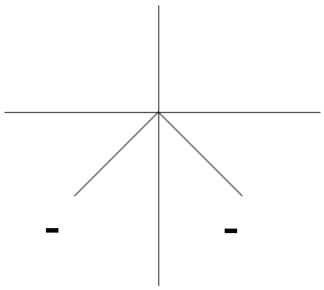
$-1 \leq ratio \leq 0$

If $\cos x = -ratio$

$-1 \leq ratio \leq 0$

If $\tan x = -ratio$

$ratio \leq 0$



$3rd: x = 180^0 + ref\angle$

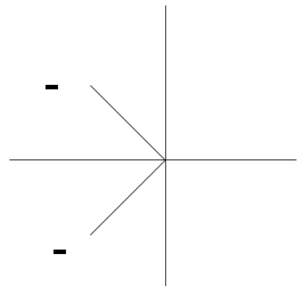
or

$4th: x = 360^0 - ref\angle$

NB: $3: x \neq 270^0 - ref\angle$

$4th: x \neq 270^0 + ref\angle$

$270^0 - ref\angle$ is in the 3rd quad but **it is not advisable to** use 270^0 as our ratios are changing to their co-ratios



$2nd: x = 180^0 - ref\angle$

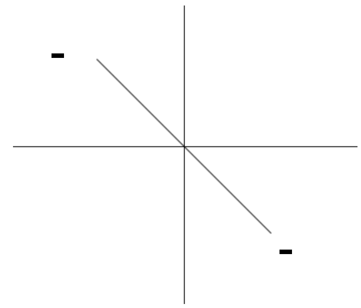
or

$3rd: x = 180^0 + ref\angle$

NB: $2nd: x \neq 90^0 + ref\angle$

$3rd: x \neq 270^0 - ref\angle$

$90^0 + ref\angle$ is in the 2nd quad but **it is not advisable to** use as our ratios are changing to their co-ratios when reducing by 90^0 as well as 270^0



$2nd: x = 180^0 - ref\angle$

or

$4th: x = 360^0 - ref\angle$

NB: $2nd: x \neq 90^0 + ref\angle$

$4th: x \neq 270^0 + ref\angle$

$270^0 + ref\angle$ is in the 3rd quad but **it is not advisable to** use 270^0 as our ratios are changing to their co-ratios as well as 90^0

Solve for *x*: ***Solutions***

1.1 $\cos x = \frac{2}{3}$

$$\cos x = \frac{2}{3}$$

$$ref\,\angle = \cos^{-1}\left(\frac{2}{3}\right)$$

$$= 48,19^0$$

$$1st : x = ref\,\angle = 48,19^0$$

$$4th : x = 360^0 - ref\,\angle = 311,81^0$$

$$\cos x = \frac{2}{3}$$

$$x = \pm\cos^{-1}\left(\frac{2}{3}\right)$$

$$= \pm 48,19^0 + 360.k,\quad k \in \mathbb{Z}$$

$$x = -311,81^0 \; or -48,19^0 \; or \; 48,19^0 \; or \; 311,81^0$$

$$\therefore x = 48,19^0 \; or \; 311,81^0$$

1.2 $\sin x = -\frac{1}{2}$

$$\sin x = -\frac{1}{2}$$

$$ref\,\angle = \sin^{-1}\left(\frac{1}{2}\right)$$

$$= 30^0$$

$$3rd : x = 180^0 + 30^0$$

$$= 210^0$$

$$4th : x = 360^0 - 30^0$$

$$x = 330^0$$

When determining the reference angle, **ignore the negative sign** to the ratio as you will get negative angle which will not preserve a reference angle.

GENERAL SOLUTIONS

note that *k* element of intergers

In determining general solutions, we do the same way as solving equations but consider the period of sine and cosine graphs as they repeat their shapes after a period of 360 and tan graph repeating itself after a period of 180.

$\sin\theta = t, 0 \le t \le 1$	$\cos\theta = t, 0 \le t \le 1$	$\tan\theta = t,$
$1: \theta = ref\,\angle + 360^0.k$	$1: \theta = ref\,\angle + 360^0.k$	$1: \theta = ref\,\angle + 180^0.k,\ k \in \mathbb{Z}$
or	or	
$2: \theta = 180^0 - ref\,\angle + 360.k, k \in \mathbb{Z}$	$4: \theta = 360^0 - ref\,\angle + 360.k, k \in \mathbb{Z}$	
$\sin\theta = t, -1 \le t \le 0$	$\cos\theta = t, -1 \le t \le 0$	$\tan\theta = t,$
$3^{rd}: \theta = 180^0 + ref\,\angle + 360^0.k$	$1: \theta = 180^0 - ref\,\angle + 360^0.k$	$1: \theta = 180^0 - ref\,\angle + 180^0.k\ \ , k \in \mathbb{Z}$
or	or	
$4^{th}: \theta = 360^0 - ref\,\angle + 360.k, k \in \mathbb{Z}$	$2: \theta = 180^0 + ref\,\angle + 360.k,\ k \in \mathbb{Z}$	

TAKE NOTE OF THESE GENERAL TYPES OF EQUATIONS

Determine the general solutions for the following equations by considering the following w**orked-out examples:**

1 $\sin\theta = \frac{1}{2}$

2 $2\sin\theta = 3\cos\theta$

3 $\cos\theta = \cos(60^0 - \alpha)$

4 $\sin\theta = \cos\alpha$

5.1 $2\cos\theta = \sin^2\theta - 2$

5.2 $\cos^2 x + \sin 2x - 1 = 0$

5.3 $\cos\theta - \sqrt{3}\sin\theta = \sqrt{3}$ and $\theta \in \left[-810^0; -540^0\right]$

1. $\sin\theta = \frac{1}{2}$ $\theta = 30^0 + 360^0.k$ **OR** $\theta = 180^0 - 30^0 + 360.k, \quad k \in \mathbb{Z}$	We are used in this type of equation from grade 10. The only thing that is new in grade 11 is general solution which has been already explained in **page 14.**
2. $2\sin\theta = 3\cos\theta$ $\sin\theta = \frac{3}{2}\cos\theta$ $\tan\theta = \frac{3}{2}$ $\theta = 56,31^0 + 180^0.k, \quad k \in \mathbb{Z}$	**NB: when trigonometric functions are not the same but angles the same. Then, divide both sides by** $\cos\theta$ **to get** $\tan\theta$**. Do not divide by** $\sin\theta$ **as you will get** $\frac{\cos\theta}{\sin\theta}$**.**
3. $\cos\theta = \cos(60^0 - \alpha)$ $\therefore \quad \theta = \pm\left(60^0 - \alpha\right) + 360^0.k, \quad k \in \mathbb{Z}$	**Since the functions are the same drop down the angles.**
4. $\sin\theta = \cos\alpha$ $\sin\theta = \sin\left(90^0 - \alpha\right)$ $ref\ \angle\theta = 90^0 - \alpha$ $\theta = 90^0 - \alpha + 360^0.k, \quad k \in \mathbb{Z}$ OR $\theta = 180^0 - (90^0 - \alpha) + 360.k$ $= 90^0 + \alpha + 360^0.k$	Since angles are not the same, we cannot divide by cosine on both sides to get a tangent, which angle will tangent be taking between the 2? Then introduction of co-functions will be applicable to make the the functions to be the same.

5. **NOTE**: If an equation does not look like 1-4 type and contains more than 2 terms.

OR

Use identities and factorise.

5.1	$2\cos\theta = \sin^2\theta - 2$ $-(1-\cos^2\theta) + 2\cos\theta + 2 = 0$ $-1+\cos^2\theta + 2\cos\theta + 2 = 0$ $\cos^2\theta + 2\cos\theta + 1 = 0$ $(\cos\theta + 1)^2 = 0$ $\cos\theta = -1$ $\theta = \pm 180^0 + 360^0.k, \quad k \in \mathbb{Z}$	• Terms more than 2, then 1-4 types not applicable • $\sin^2\theta$ can be written in the terms of $\cos\theta$ using square identities, e.g., $\sin^2\theta = 1-\cos^2\theta$.
5.2	$\cos^2 x + \sin 2x - 1 = 0$ $\cos^2 x + 2\sin x\cos x - 1 = 0$ $\cos^2 x + 2\sin x\cos x - (\cos^2 x + \sin^2 x) = 0$ $\cos^2 x + 2\sin x\cos x - \cos^2 x - \sin^2 x = 0$ $2\sin x\cos x - \sin^2 x = 0$ $\sin x(2\cos x - \sin x) = 0$ $\sin x = 0 \quad or \quad 2\cos x - \sin x = 0$ $x = 0^0 + 360^0.k \;, \quad k \in \mathbb{Z}$ *OR* $2\cos x - \sin x = 0$ $\sin x = 2\cos x$ $\tan x = 2$ $x = 63,44^0 + 180^0.k$	• Terms more than 2, then 1-4 types not applicable • Change of double angle to single angle as $\sin 2x = 2\sin x\cos x$ • Since we have $\sin x$ **and** $\cos x$ we need the identity of 1 in terms of $\sin x$ **and** $\cos x$. $1 = \cos^2 x + \sin^2 x$ • Simplification will lead to 2 terms, then factorise • $\sin x = 2\cos x$ functins not the same but ratios the same then divide by $\cos x$ on both sides to get $\tan x$ on the left hand side.

5.3 Solve for ***θ*** if $\cos\theta-\sqrt{3}\sin\theta=\sqrt{3}$ and $\theta\in\left[-810^0;-540^0\right]$

$$\cos\theta-\sqrt{3}\sin\theta=\sqrt{3}$$
$$\cos\theta=\sqrt{3}+\sqrt{3}\sin\theta$$
$$\cos^2\theta=3+6\sin\theta+3\sin^2\theta$$
$$1-\sin^2\theta=3\sin^2\theta+6\sin\theta+3$$
$$4\sin^2\theta+6\sin\theta+2=0$$
$$2\sin^2\theta+3\sin\theta+1=0$$
$$(2\sin\theta+1)(\sin\theta+1)=0$$
$$\sin\theta=-\frac{1}{2}\ or\ \sin\theta=-1$$
$$\theta=210^0+360^0.k\ \ or\ \ \theta=270^0+360^0.k,\ \ k\in\mathbb{Z}$$
$$or$$
$$\theta=330^0+360^0.k$$
$$If\ k=-2\ or\ -3$$
$$then$$
$$\therefore\ \theta=-510^0\ \ or-450^0\ \ or-390^0\ \ or-750^0\ \ or\ -810^0$$

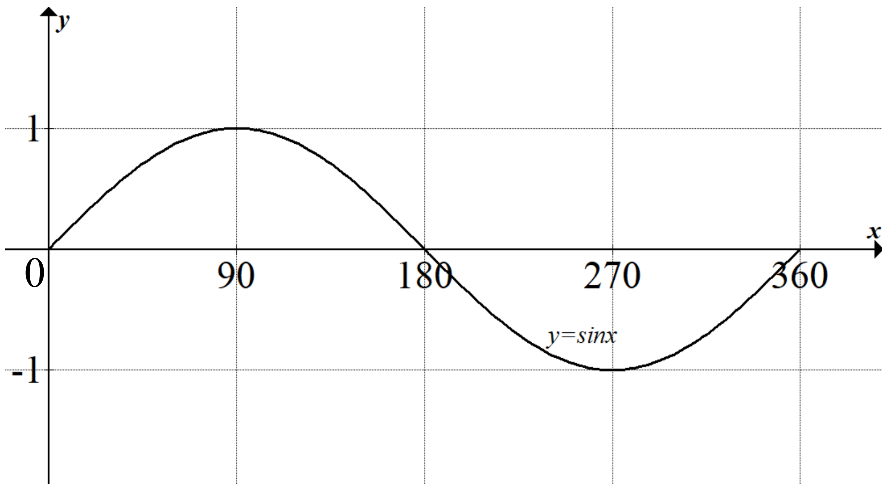
- Terms more than 2, then 1-4 types not applicable
- Take all terms having square root to the same side.
- Then, square both sides of the equation and also show squaring on your calculations
- Write equation in its standard form
- Factorise
- Since the interval is $\theta\in\left[-810^0;-540^0\right]$ **for values of θ, *k*-values must be** -3 to -2

TRIGONOMETRIC GRAPHS

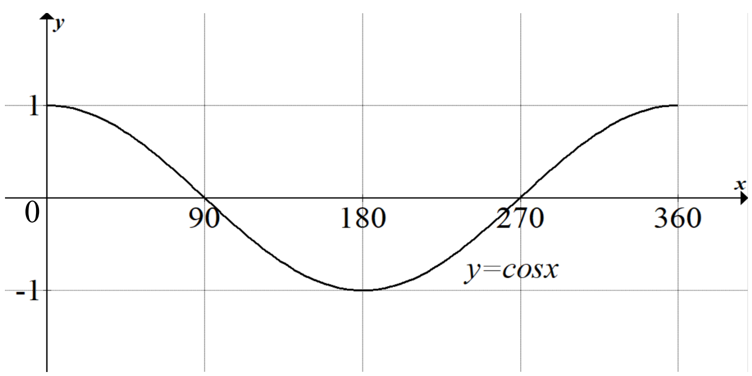
Basic trigonometric graphs/functions of sine and cosine have same characteristics except their shapes.

3 basic/mother trigonometric graphs/functions are shown below:

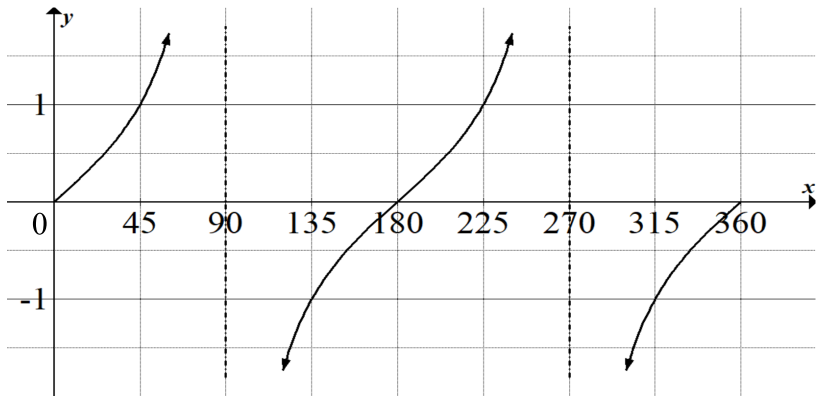
$y = \sin x, \quad 0^0 \le x \le 360^0$



$y = \cos x, \quad 0^0 \le x \le 360^0$



$y = \tan x, \quad 0^0 \leq x \leq 360^0$

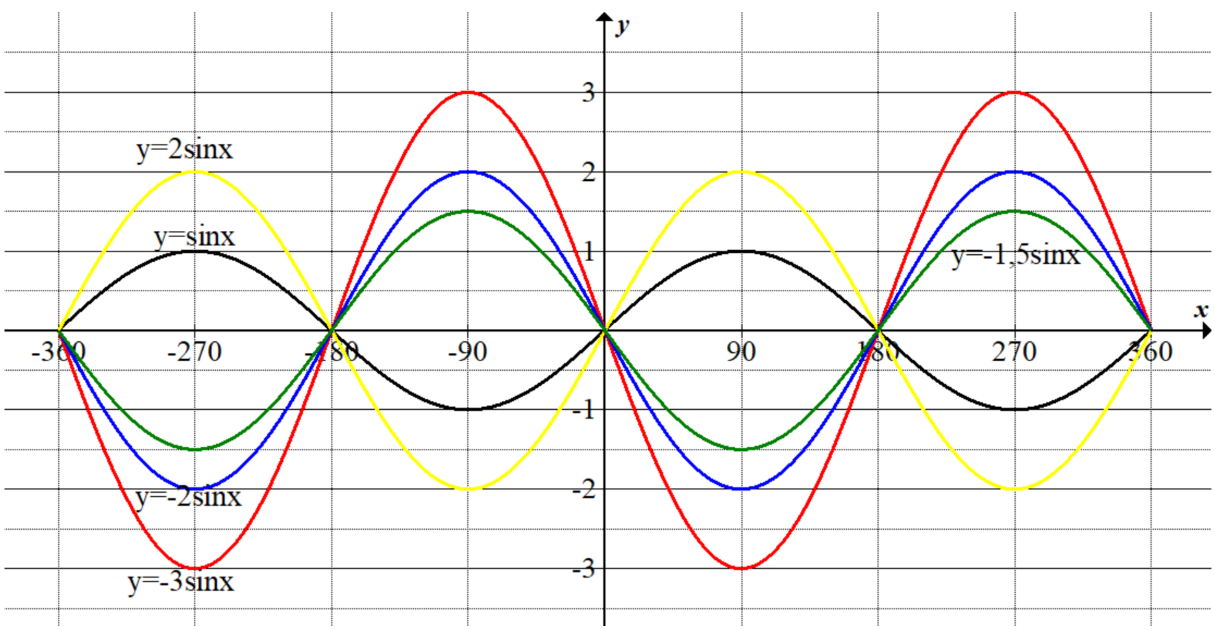


NB You need to be able to sketch, recognise and interpret graphs of the following:

- $y = a\sin k(x+p)+q$
- $y = a\cos k(x+p)+q$
- $y = a\tan k(x+p)+q$

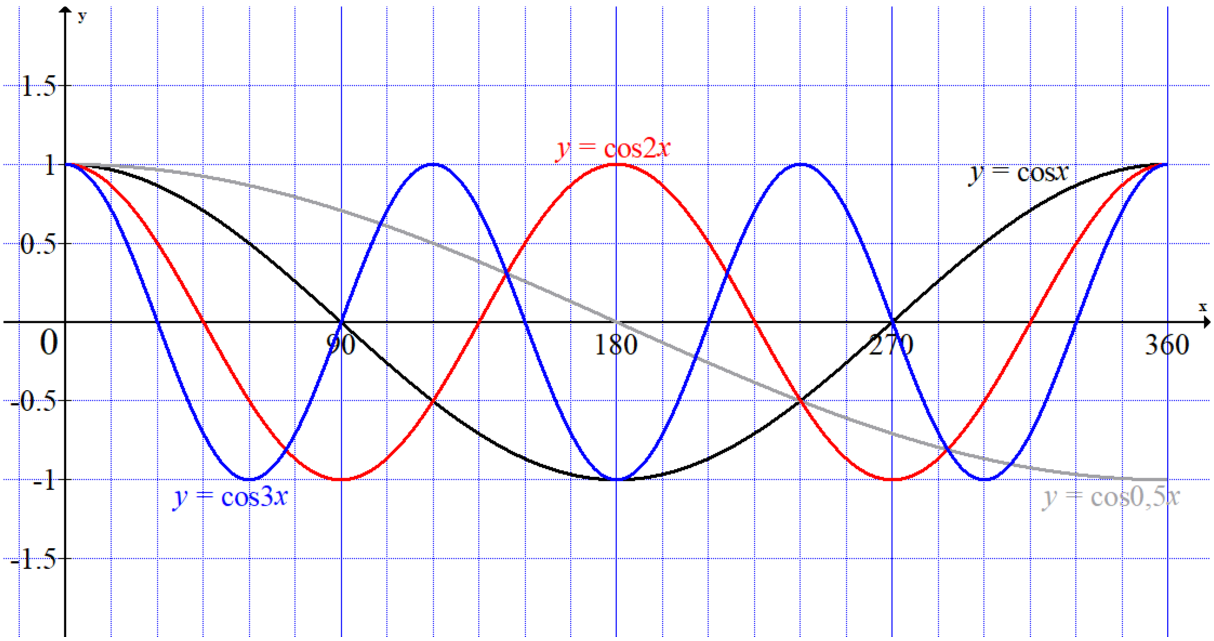
Observe the effects of *a*, *k, p* and *q* on the basic graphs as shown below

Effects of *a*



***a* affects the amplitude of sine and cosine graphs. If $a < 0$ the basic graph flips along the *x*-axis,**

Effects of *k*



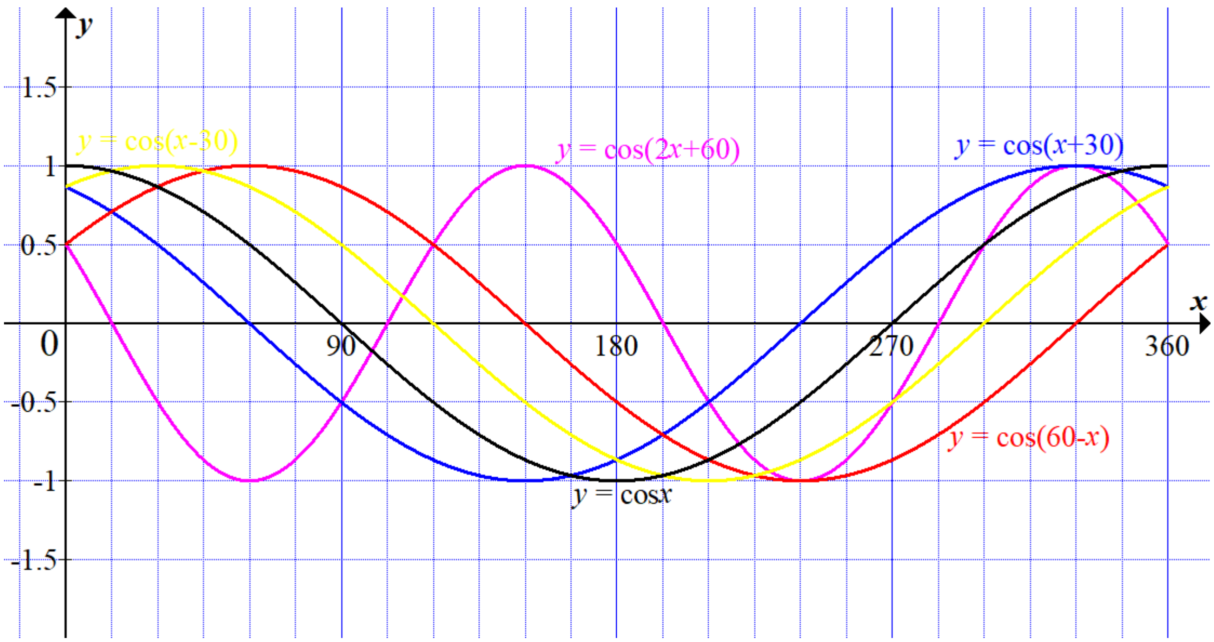
k indicates the contraction or expansion of the graph.

k **affects the period of the graph** in the following way:

For sine and cosine graphs, the period becomes $\frac{360^0}{k}$

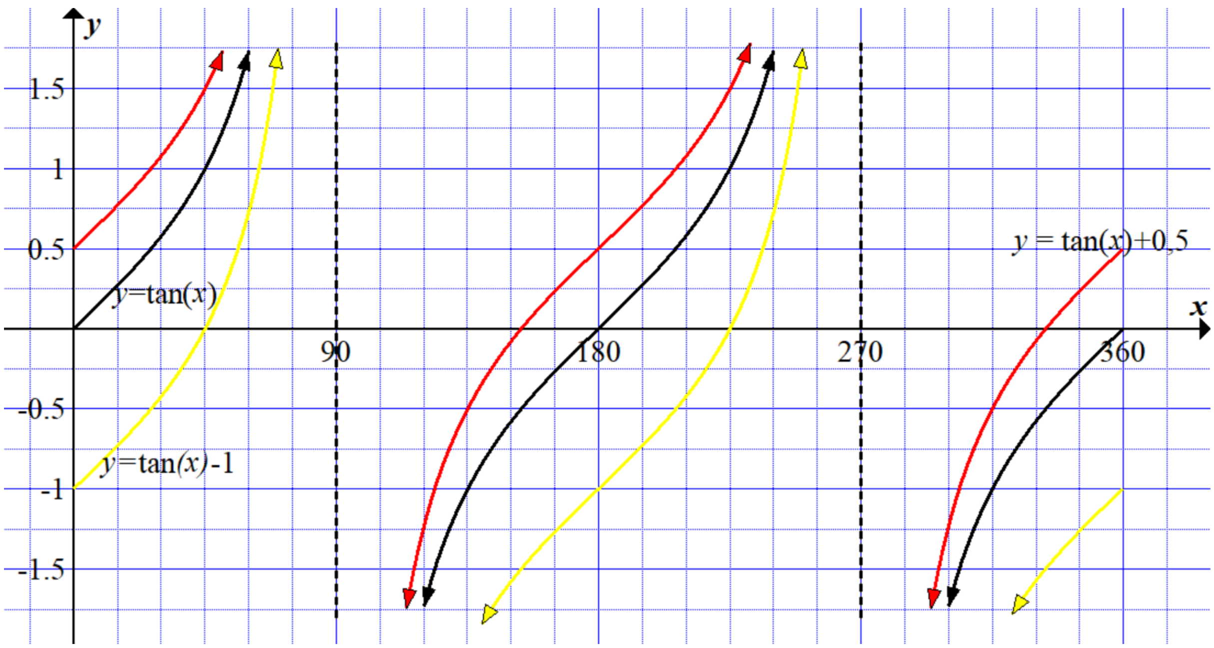
The period of the tangent graph is $\frac{180^0}{k}$.

Effects of *p*



p shifts the graphs horizontally (4 graphs shifting)

Effects of *q*



q shifts the graph vertically

BASIC PROPERTIES OF TRIGONOMETRIC GRAPHS

Worked-out Example 1

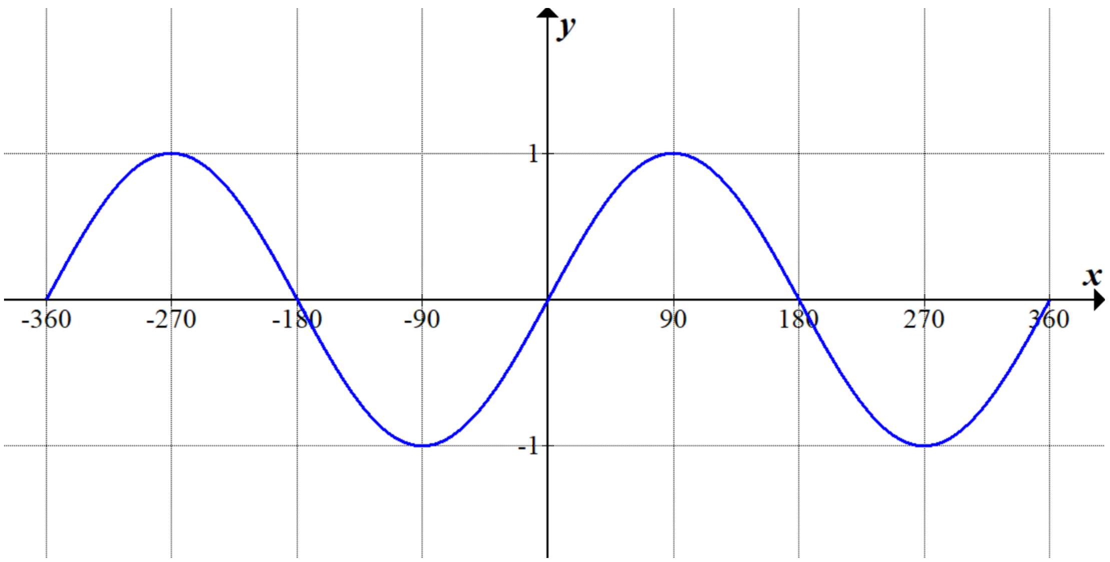
	Period	Amplitude	Range
$y = \sin x$	360^0	$\frac{1}{2}[1-(-1)]=1$	$-1 \le y \le 1$ or $y \in [-1;1]$
$y = \cos x$	360^0	$\frac{1}{2}[1-(-1)]=1$	$-1 \le y \le 1$ or $y \in [-1;1]$
$y = \tan x$	180^0	tangent doesn't have a min/max *y*-value. $\therefore$ amplitude not available	$y \in R$

Worked-out Example 2

Trigonometric graph	Period	Amplitude	Range
$y = \frac{1}{2}\sin\theta$	360^0	$\frac{1}{2}$	$-\frac{1}{2} \le y \le \frac{1}{2}$ *or* $y \in \left[-\frac{1}{2} ; \frac{1}{2}\right]$
$y = -3\sin 2\theta$	$\frac{360^0}{2} = 180^0$	3	$-3 \le y \le 3$ *or* $y \in [-3 ; 3]$
$y = \cos\frac{1}{2}\theta$	$\frac{360^0}{\frac{1}{2}} = 720^0$	1	$-1 \le y \le 1$ *or* $y \in [-1 ; 1]$
$y = 4\cos\frac{3}{4}\theta$	$\frac{360}{\frac{3}{4}} = 480^0$	4	$-4 \le y \le 4$ *or* $y \in [-4 ; 4]$

Worked-out Example 1

1. Use the sine graph given below to answer the following questions:



1.1 What are the minimum and maximum values of $y = \sin x$? (2)

1.2 What is the domain and range of $y = \sin x$ (4)

1.3 Write down the *x*-intercepts of $y = \sin x$ (2)

1.4 What is the amplitude and period of $y = \sin x$? (2)

Solutions

1.1 Minimum value $= -1$ ✓and maximum value $= 1$ ✓ (2)

1.2 Domain : $x \in [-360^0;360^0], x \in R$ ✓✓ Range: $[-1;1] y \in R$ ✓✓ (4)

1.3 *x*-intercepts: $-360^0;-180^0;0^0;180^0;360^0$. ✓✓ (2)

1.4 Amplitude is1 and period 360^0 .✓✓ (2)

Worked-out Example 2

2. Consider a function $g(x) = -\cos x + 1$

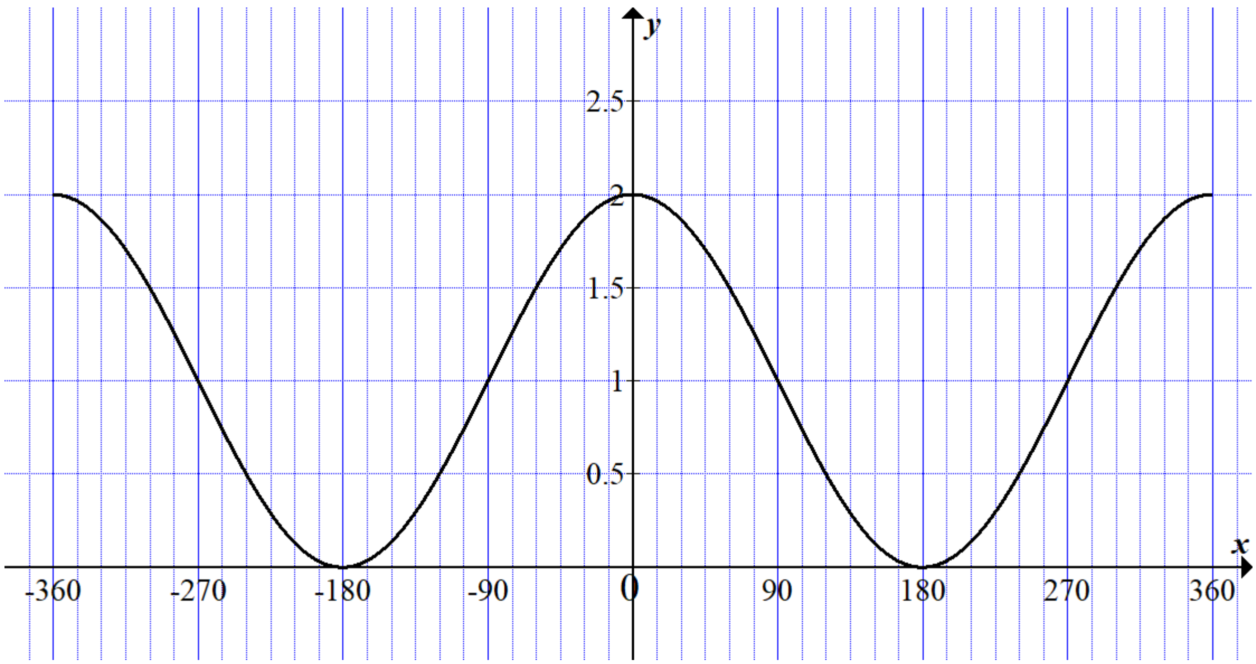
2.1 sketch the graph of *g* for $x \in [-360^0; 360^0]$

2.2 write down the period and amplitude of *g*

2.3 write down the range of *g*

Solutions

2.1



2.2 period: 360^0 amplitude: 1

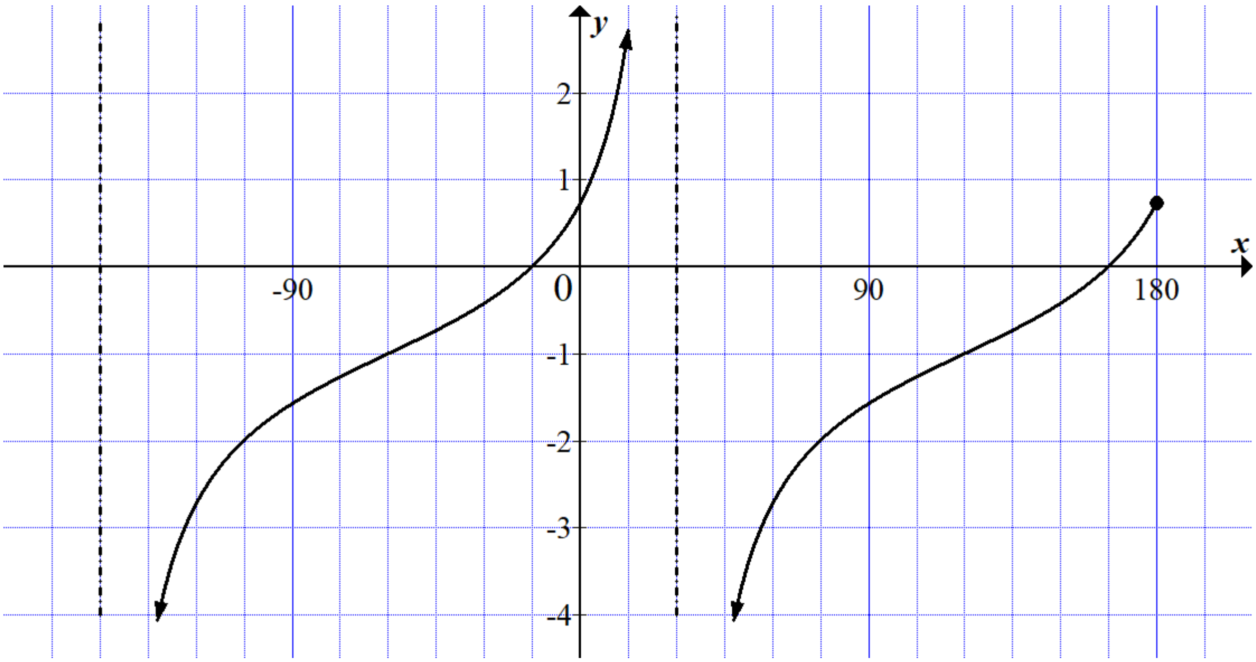
2.3 Range: $y \in [0;2]$ OR $0 \leq y \leq 2$

Worked-out Example 3

3.1 Sketch a graph of $y = \tan(x + 60^0) - 1; x \in [-150^0; 180^0]$

Solution

3.1



Worked-out Example 4

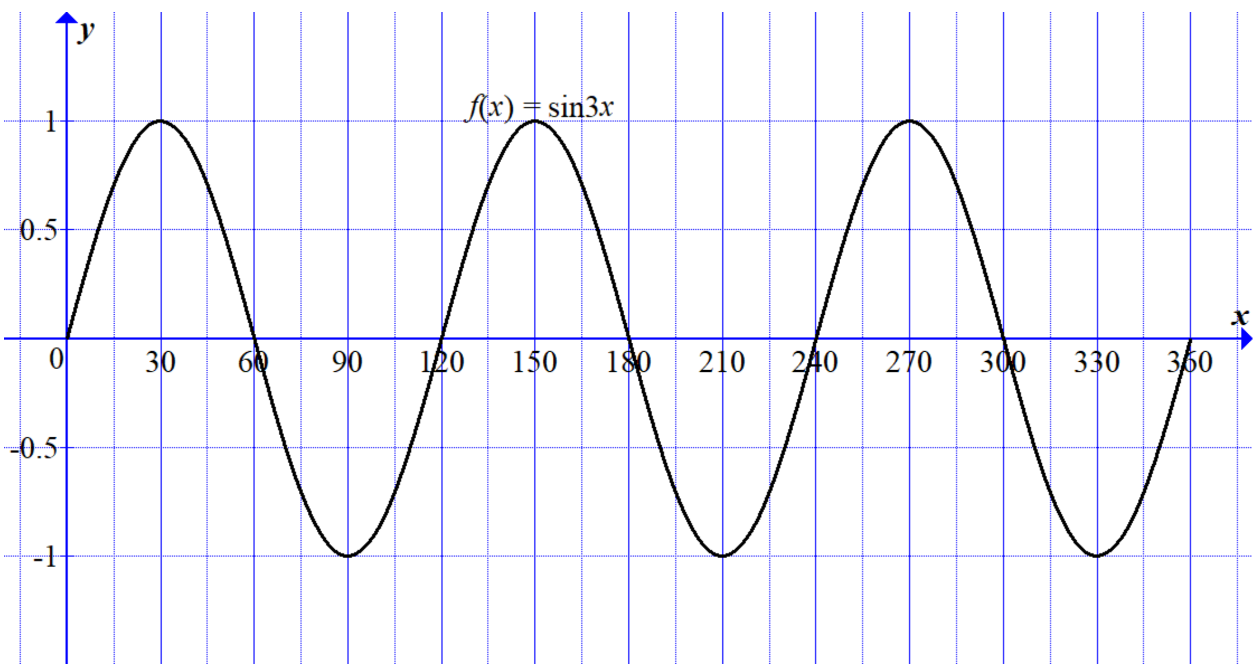
4.1 Sketch the graph of $f(x) = \sin 3x,\ x \in [0^0\ ;\ 360^0]$

Some thought process:

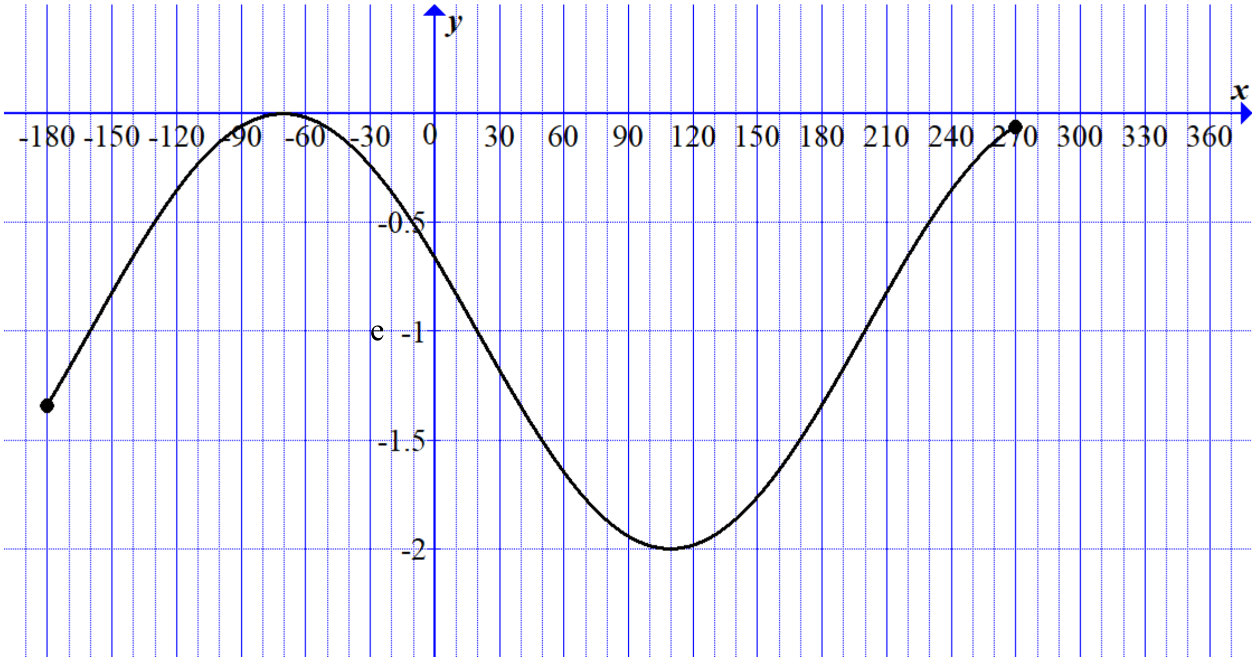
- This is a sine graph with $k = 3$
- The period will be $\dfrac{360^0}{3} = 120^0$
- For drawing our graph, we can divide all the *x*-values on the standard sin graph by 3. That will mean our standard *x*-values when using table method: $\{0^0;90^0;180^0;270^0;360^0\}\ x \in [0^0;360^0]$ will for the new graph now be: $\{0^0;30^0;60^0;90^0;120^0;...\}\ x \in [0^0;360^0]$
- Finding table using Casio *fx* calculator:
  Step 1: click on mode
  Step 2: select table
  Step 3: type sin 3x
  Step 4: Start at 0^0 and end at 360^0 since $x \in [0^0;360^0]$
  Step 5: Step by period divide by number of quadrants

Final sketch

4.1



Worked-out Example 5



- The graph has been reflected, $\therefore -\text{sine}$ graph, thus $a = -1$
- The graph has shifted 20^0 to the right, $p = -20^0$
- Middle *y*-value is -1
- Amplitude is $\frac{1}{2}[0-(-2)] = 1$
- The equation is therefore: $y = -\sin(x - 20^0) - 1$

GRAPHICAL INTERPRETATION

Worked-out Example 1

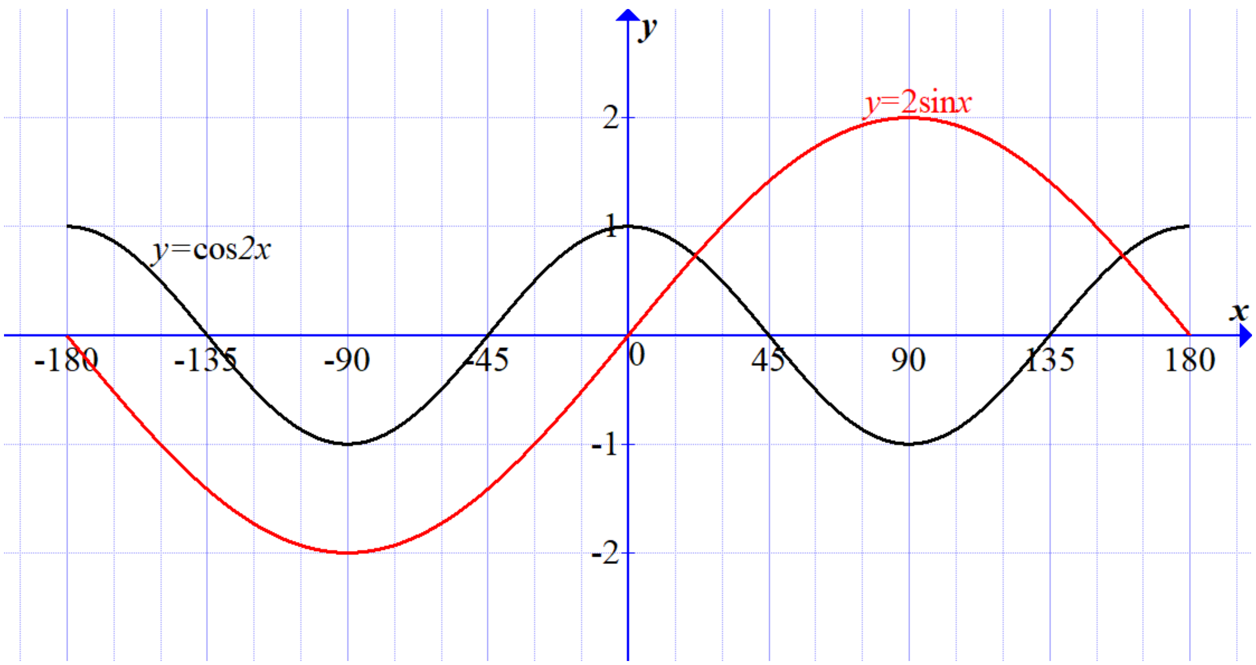
1. Sketch the graphs of: $y = 2\sin x$ and $y = \cos 2x$ if $-180^0 \leq x \leq 180^0$ on the same system of axes.

1.1 For which value(s) of *x* is $2\sin x > 0$ **?**

1.2 For which value(s) of x is $\frac{1}{2}\cos 2x - \sin x = 0$

Solutions

1.



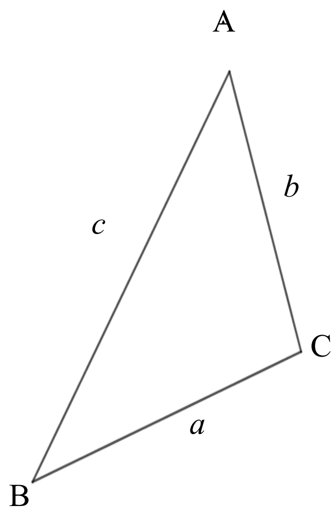
1.1 $0^0 < x < 180^0$

1.2

$$\frac{1}{2}\cos 2x - \sin x = 0$$
$$\cos 2x = 2\sin x$$
$$1 - 2\sin^2 x = 2\sin x$$
$$2\sin^2 x + 2\sin x - 1 = 0$$
$$\sin x = \frac{-2 \pm \sqrt{(2)^2 - 4(2)(-1)}}{2(2)}$$
$$\sin x \neq \frac{-2 - \sqrt{12}}{4}$$
$$or$$
$$\sin x = \frac{-2 + \sqrt{12}}{4}$$
$$ref\angle = 21,47^0$$
$$x = 21,27^0 \quad or \quad x = 158,73^0$$

SOLVING 2D AND 3D PROBLEMS

In any triangle:



$$\text{Sine rule}: \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$
$$\text{Cosine rule}: a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$\text{Area rule}: A.\Delta ABC = \frac{1}{2}ab\sin C$$
$$= \frac{1}{2}ac\sin B$$
$$= \frac{1}{2}bc\sin A$$

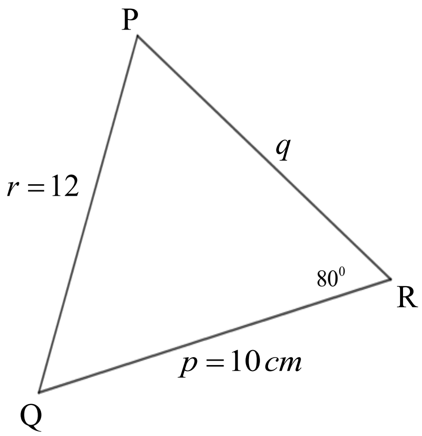
SINE RULE

- Sine rule is applicable when given two sides and an angle in any triangle, then you can be able to calculate the 2nd angle.

Worked-out Example:

In $\triangle PQR$, $PQ=12cm$, $QR=10cm$ and $\hat{R}=80^0$. Determine the:

a) size $\hat{P}$
b) length of PR



Solutions

a)

$$\frac{\sin P}{p}=\frac{\sin R}{r}$$

$$\frac{\sin P}{10}=\frac{\sin 80^0}{12}$$

$$\sin P=\frac{10\times\sin 80}{12}$$

$$\hat{P}=\sin^{-1}\left(\frac{10\times\sin 80^0}{12}\right)$$

$$\hat{P}=55.15^0$$

b) For you to be able to get the length of PR you will need to know $\hat{Q}$. Now you know two angles in ΔPQR then you can get the 3rd one by applying sum of angles in a Δ.

$$\hat{Q}=44,85^0 \; [sum\,of\,\angle s\,in\,a\,\Delta]$$

$$\frac{q}{\sin Q}=\frac{r}{\sin R}$$

$$\frac{q}{\sin 44,85^0}=\frac{12}{\sin 80^0}$$

$$q=\frac{12\times\sin 44,85^0}{\sin 80^0}$$

$$=8,59\,cm$$

- Sine rule is also applicable when given two angles and a side, then you will be able to use it to calculate the other sides as well as the 3rd angle

Worked-out Example 2

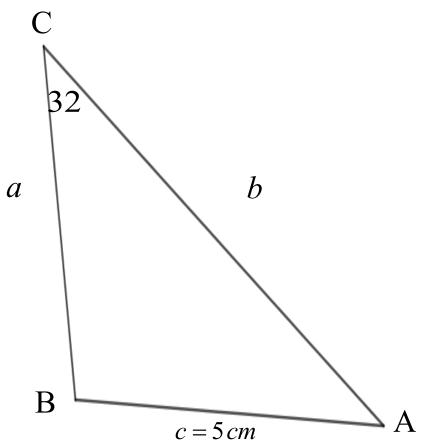
In ΔABC, $\hat{A}=50^0$, $\hat{C}=32^0$ and $AB=5\,\text{cm}$. Determine:

a) the value of *a* length of BC.

Solutio

b) the size of $\hat{B}$

c) the value *b* length of AC



a)
$$\frac{a}{\sin A}=\frac{c}{\sin C}$$
$$\frac{a}{\sin 50^0}=\frac{5}{\sin 32^0}$$
$$a=7,23$$

b) $\hat{B}=98^0$ $[sum\ of\ \angle s\ in\ a\ \Delta]$

c)
$$\frac{b}{\sin B}=\frac{c}{\sin C}$$
$$\frac{b}{\sin 98^0}=\frac{5}{\sin 32^0}$$
$$b=9,34$$

Then we know now all the angles and lengths of the sides in this triangle. You cannot use trig ratios solving this triangle, as it is not a right-angled triangle.

COSINE RULE

- Cosine rule is applicable when given length of all the 3 sides of a triangle, you can be able to calculate any angle in the triangle. The 2nd angle can be calculated by applying cosine rule or sine rule it will depend on you.

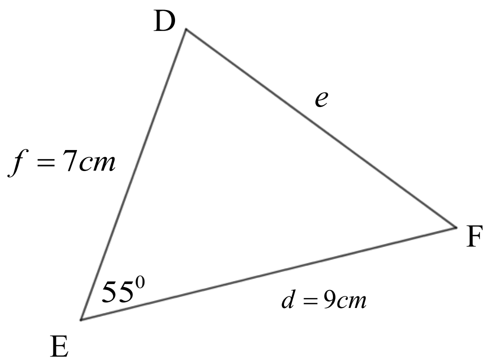
Worked-out Example 1

In $\triangle$DEF, $DE = 7\,cm,\ FE = 9\,cm\ and\ \hat{E} = 55^0$.

Determine the:

a) length of DF

b) size of $\hat{F}$



Solutions

Applying cosine with **sides and included angle**

a)

$$DF^2 = DE^2 + EF^2 - 2DE.EF.\cos\hat{E}$$
$$= (7)^2 + (9)^2 - 2(7)(9)\cos 55^0$$
$$DF = \sqrt{(7)^2 + (9)^2 - 2(7)(9)\cos 55^0}$$
$$= 7,60$$

b) To get the 2nd angle you can apply sine rule as well but for now we are going to apply cosine rule when **having all the 3 sides. Since we are looking for** $\hat{F}$, then the side opposite to $\hat{F}$ will be the subject of the formula in this way.

$$DE^2 = EF^2 + DF^2 - 2.EF.DF\cos\hat{F}$$
$$7^2 = 9^2 + (7,60)^2 - 2.9.7,60.\cos\hat{F}$$
$$\cos\hat{F} = \frac{9^2 + (7,60)^2 - 7^2}{(2)(9)(7,60)}$$
$$\hat{F} = \cos^{-1}\left(\frac{9^2 + (7,60)^2 - 7^2}{2(9)(7,60)}\right)$$
$$\hat{F} = 48,99^0$$

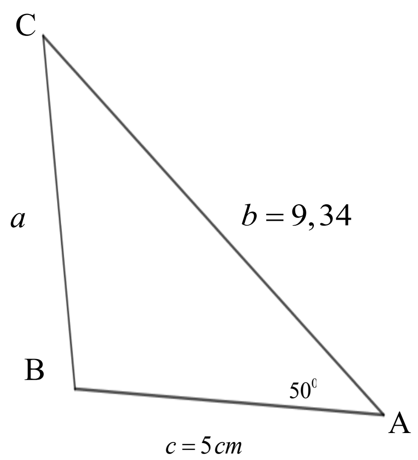
AREA RULE

- Area rule is applicable when you are given two sides and included angle, then you can calculate the area of the triangle.

Worked-out Example 1

In ΔABC, $\hat{A}=50^0$, $AC = 9,34$ and $AB = 5\,cm$.

a) Determine the area of ΔABC



a) $$A.of\ \Delta ABC = \frac{1}{2}.AC.AB\sin\hat{A}$$
$$= \frac{1}{2}\times 9,34\times 5\times \sin 50^0$$
$$= 17,89\ cm^2$$

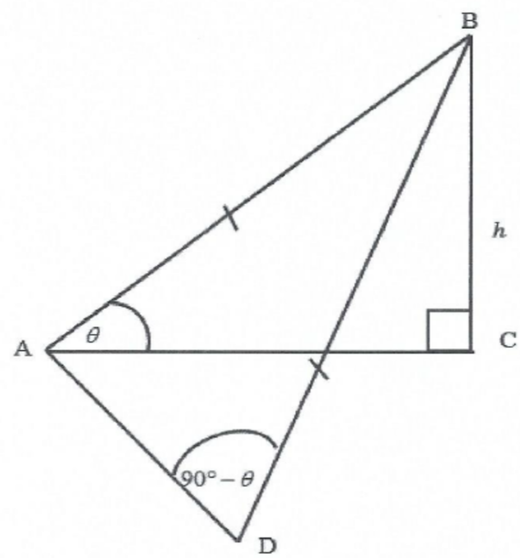
3.4 TYPICAL EXAM QUESTIONS

Example 1

Mathematics P2 9 NSC LimpopoDoE/September 2019

QUESTION 1

In the diagram below, BC is a pole anchored by two cables at A and D. A, D and C are in the same horizontal plane. The height of the pole is h and the angle of elevation from A to the top of the pole B, is θ. BA = BD and $B\hat{D}A = 90° - \theta$.



1.1 Express AB in terms of h and a trigonometric ratio of θ. (1)

1.2 Determine the magnitude of $A\hat{B}D$ in terms of θ. (3)

1.3 Determine the distance between the two anchors in terms of h. (3)
[7]

Solutions

1.1

$$In\,\Delta\,ABC: \sin\theta = \frac{h}{AB}$$

$$AB = \frac{h}{\sin\theta}$$

✓ $AB = \dfrac{h}{\sin\theta}$ (1)

1.2 $In\,\Delta\,ABD: B\hat{A}D = \hat{D} = 90^0 - \theta$ (sides opp $= \angle s$)

$90^0 - \theta + 90^0 - \theta + A\hat{B}D = 180^0$ ($\angle s$ of Δ)

$A\hat{B}D = 2\theta$

✓ $B\hat{A}D = \hat{D} = 90^0 - \theta$ ✓Reason (3)

1.3

✓ $A\hat{B}D = 2\theta$

7.3

$$\frac{AD}{\sin 2\theta} = \frac{AB}{\sin(90^0 - \theta)}$$

$$AD = \frac{\frac{h}{\sin\theta} \times \sin 2\theta}{\cos\theta}$$

$$AD = \frac{\frac{h}{\sin\theta} \times 2\sin\theta\cos\theta}{\cos\theta}$$

$$AD = 2h$$

✓ $\dfrac{AD}{\sin 2\theta} = \dfrac{AB}{\sin(90^0 - \theta)}$ (3)

✓ $\cos\theta$

✓ $2\sin\theta\cos\theta$

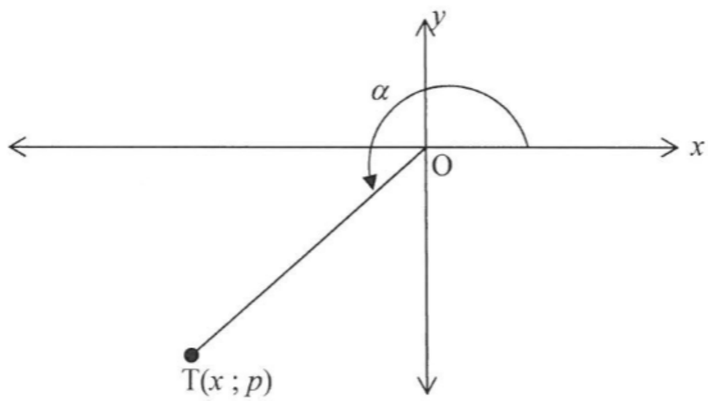
QUESTION 2

2.1 If $x = 3\sin\theta$ and $y = 3\cos\theta$, determine the value of $x^2 + y^2$. (3)

2.2 Simplify to a single term:

$\sin(540^\circ - x).\sin(-x) - \cos(180^\circ - x).\sin(90^\circ + x)$ (6)

2.3 In the diagram below, T$(x\,;\,p)$ is a point in the third quadrant and it is given that $\sin\alpha = \dfrac{p}{\sqrt{1+p^2}}$.



2.3.1 Show that $x = -1$. (3)

2.3.2 Write $\cos(180^\circ + \alpha)$ in terms of p in its simplest form. (2)

2.3.3 Show that $\cos 2\alpha$ can be written as $\dfrac{1-p^2}{1+p^2}$. (3)

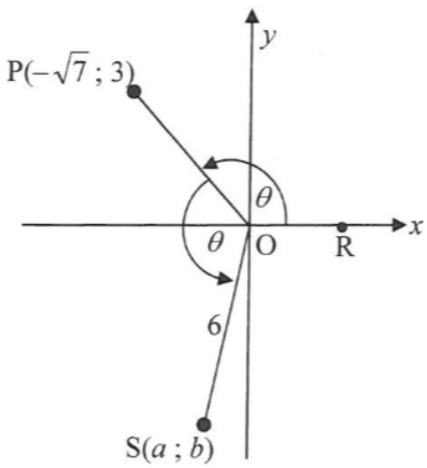
2.4 **2.4.1** For which value(s) of x will $\dfrac{2\tan x - \sin 2x}{2\sin^2 x}$ be undefined in the interval $0^\circ \leq x \leq 180^\circ$? (3)

2.4.2 Prove the identity: $\dfrac{2\tan x - \sin 2x}{2\sin^2 x} = \tan x$ (6)

[26]

QUESTION 3

3.1 $P(-\sqrt{7}\,;3)$ and $S(a\,;b)$ are points on the Cartesian plane, as shown in the diagram below. $P\hat{O}R = P\hat{O}S = \theta$ and $OS = 6$.



Determine, WITHOUT using a calculator, the value of:

3.1.1 $\tan\theta$ (1)

3.1.2 $\sin(-\theta)$ (3)

3.1.3 a (4)

3.2 **3.2.** Simplify $\dfrac{4\sin x\cos x}{2\sin^2 x-1}$ to a single trigonometric ratio. (3)

3.2.2 Hence, calculate the value of $\dfrac{4\sin 15°\cos 15°}{2\sin^2 15°-1}$ WITHOUT using a calculator. (Leave your answer in simplest surd form.) (2)

[13]

QUESTION 4

4.1 **Without using a calculator**, determine the following in terms of $\sin 36°$:

4.1. $\sin 324°$ (1)

4.1.2 $\cos 72°$ (2)

4.2 Prove the identity: $1-\dfrac{\tan^2\theta}{1+\tan^2\theta}=\cos^2\theta$ (4)

4.3 Use QUESTION 6.2 to determine the general solution of:

$$1-\frac{\tan^2\frac{1}{2}x}{1+\tan^2\frac{1}{2}x}=\frac{1}{4}$$

(6)

4.4 Given: $\cos(A-B)=\cos A\cos B+\sin A\sin B$

4.4.1 Use the formula for $\cos(A-B)$ to derive a formula for $\sin(A-B)$. (4)

4.4.2 **Without using a calculator**, show that

$$\sin(x+64°)\cos(x+379°)+\sin(x+19°)\cos(x+244°)=\frac{1}{\sqrt{2}}$$

for all values of x. (6)

[23]

QUESTION 5

5.1 If $\cos 2\theta = -\frac{5}{6}$, where $2\theta \in [180°;270°]$, calculate, **without using a calculator**, the values in simplest form of:

5.1.1 $\sin 2\theta$ (4)

5.1.2 $\sin^2 \theta$ (3)

5.2 Simplify $\sin(180° - x).\cos(-x) + \cos(90° + x).\cos(x - 180°)$ to a single trigonometric ratio. (6)

5.3 Determine the value of $\sin 3x.\cos y + \cos 3x.\sin y$ if $3x + y = 270°$. (2)

5.4 Given: $2\cos x = 3\tan x$

5.4.1 Show that the equation can be rewritten as $2\sin^2 x + 3\sin x - 2 = 0$. (3)

5.4.2 Determine the general solution of x if $2\cos x = 3\tan x$. (5)

5.4.3 Hence, determine two values of y, $144° \leq y \leq 216°$, that are solutions of $2\cos 5y = 3\tan 5y$. (4)

5.5 Consider: $g(x) = -4\cos(x + 30°)$

5.5.1 Write down the maximum value of $g(x)$. (1)

5.5.2 Determine the range of $g(x) + 1$. (2)

5.5.3 The graph of g is shifted $60°$ to the left and then reflected about the x-axis to form a new graph h. Determine the equation of h in its simplest form. (3)

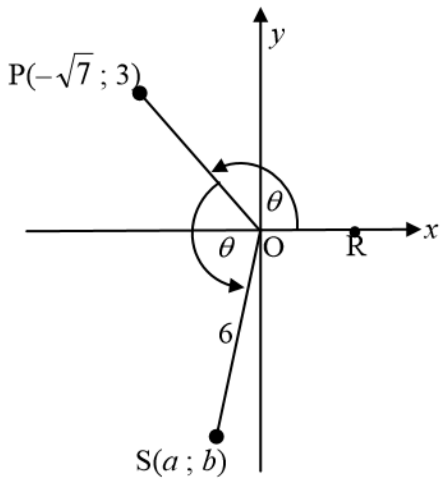
[33]

QUESTION 2

2.1	$x^2 + y^2$ $=(3\sin\theta)^2 + (3\cos\theta)^2$ $=9\sin^2\theta + 9\cos^2\theta$ $=9(\sin^2\theta + \cos^2\theta)$ $=9(1)$ $=9$	✓ simpl/*vereenv* ✓ CF/*GF* = 9 ✓ answer/*antw* (3)
2.2	$\sin(540° - x).\sin(-x) - \cos(180° - x).\sin(90° + x)$ $\sin(180° - x).\sin(-x) - \cos(180° - x).\sin(90° + x)$ $= (\sin x)(-\sin x) - (-\cos x)(\cos x)$ $= -\sin^2 x + \cos^2 x$ $= \cos 2x$	✓ $\sin(540° - x) = \sin x$ ✓ $\sin(-x) = -\sin x$ ✓ $\cos(180° - x) = -\cos x$ ✓ $\sin(90° + x) = \cos x$ ✓ $-\sin^2 x + \cos^2 x$ ✓ $\cos 2x$ (6)

2.3.1	$OT = \sqrt{x^2 + p^2}$ $\sin\alpha = \frac{y_T}{OT}$ $= \frac{p}{\sqrt{x^2 + p^2}}$ $\frac{p}{\sqrt{x^2 + p^2}} = \frac{p}{\sqrt{1 + p^2}}$ $x^2 = 1$ $x = -1$ **OR/*OF*** (P lies in 3rd quadrant) $x^2 + y^2 = r^2$ $x^2 + p^2 = \left(\sqrt{1 + p^2}\right)^2$ $x^2 + p^2 = 1 + p^2$ $x^2 = 1$ $x = -1$ (P lies in 3rd quadrant)	✓ $OT = \sqrt{x^2 + p^2}$ ✓ $\sin\alpha = \frac{y_T}{OT}$ ✓ $x^2 = 1$ (3) ✓ $x^2 + y^2 = r^2$ ✓ subst ✓ $x^2 = 1$ (3)
2.3.2	$\cos(180° + \alpha)$ $= -\cos\alpha$ $= -\left(\frac{-1}{\sqrt{1 + p^2}}\right)$ $= \frac{1}{\sqrt{1 + p^2}}$	✓ $-\cos\alpha$ ✓ answer/*antw* (2)

QUESTION 3



3.1.	$\tan\theta = -\dfrac{3}{\sqrt{7}}$	✓answ/*antw* (1)
3.1.2	$\sin(-\theta) = -\sin\theta$ $OP^2 = \left(-\sqrt{7}\right)^2 + 3^2$ $OP^2 = 16$ $OP = 4$ $\sin(-\theta) = -\dfrac{3}{4}$	✓reduction/ *reduksie* ✓ OP = 4 ✓answ/*antw* (3)
3.1.	$\dfrac{a}{6} = \cos 2\theta$ $a = 6(1 - 2\sin^2\theta)$ $= 6 - 12\left(\dfrac{3}{4}\right)^2$ $= \dfrac{24}{4} - \dfrac{27}{4}$ $= -\dfrac{3}{4}$ **OR/*OF***	✓trig ratio/*verh* ✓expansion/ *uitbreiding* ✓ $\sin\theta = \dfrac{3}{4}$ ✓answ/*antw* (4)

3.1.3	$\frac{a}{6} = \cos 2\theta$ $a = 6(2\cos^2\theta - 1)$ $= 12\left(\frac{-\sqrt{7}}{4}\right)^2 - 6$ $= \frac{21}{4} - \frac{24}{4}$ $= -\frac{3}{4}$ **OR/*OF***	✓trig ratio/*verh* ✓expansion/ *uitbreiding* ✓ $\cos\theta = \frac{-\sqrt{7}}{4}$ ✓answ/*antw* (4)
3.2.1	$\frac{4\sin x.\cos x}{2\sin^2 x - 1} = \frac{2(2\sin x.\cos x)}{-(1-2\sin^2 x)}$ $= \frac{2\sin 2x}{-\cos 2x}$ $= -2\tan 2x$	✓2sin $2x$ ✓–cos $2x$ ✓answ/*antw* (3)
3.2.2	$\frac{4\sin 15°\cos 15°}{2\sin^2 15° - 1} = -2\tan 2(15°)$ $= -2\tan 30°$ $= -2\left(\frac{1}{\sqrt{3}}\right)$ $= -\frac{2}{\sqrt{3}}$ **OR/*OF*** $-\frac{2\sqrt{3}}{3}$	✓ $-2\tan 2(15°)$ ✓answ/*antw* (2) **[13]**

QUESTION 4

4.1.	$\sin(360° - 36°) = -\sin 36°$	✓ answer (1)
4.1.	$\cos 72° = \cos(2 \times 36°)$ $= 1 - 2\sin^2 36°$ Answer only: Full marks	✓ double angle/*dubbelhoek* ✓answer (2)

4.2	R.T.P.: $1 - \dfrac{\tan^2\theta}{1+\tan^2\theta} = \cos^2\theta$ $\text{LHS} = \dfrac{1+\tan^2\theta - \tan^2\theta}{1+\tan^2\theta}$ $= \dfrac{1}{1+\dfrac{\sin^2\theta}{\cos^2\theta}}$ $= \dfrac{1}{\dfrac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}}$ $= \dfrac{1}{\dfrac{1}{\cos^2\theta}}$ $= \cos^2\theta$ = RHS **OR/*OF***	✓ writing as a single fraction/*skryf as enkelbreuk* ✓ quotient identity/ *kwosiëntidentiteit* ✓ denominator as a single fraction / *Noemer as enkelbreuk* ✓ square identity/*vierkantidentiteit* (4)

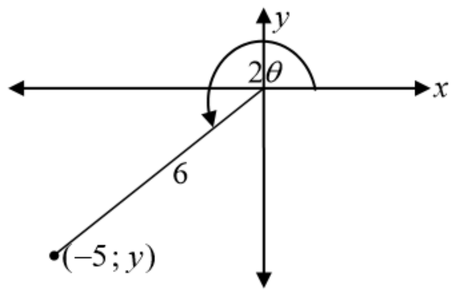
4.2	$\text{LHS}=\dfrac{1+\tan^2\theta-\tan^2\theta}{1+\tan^2\theta}$ $=\dfrac{1}{1+\dfrac{\sin^2\theta}{\cos^2\theta}}$ $=\dfrac{1}{1+\dfrac{\sin^2\theta}{\cos^2\theta}}\times\dfrac{\cos^2\theta}{\cos^2\theta}$ $=\dfrac{\cos^2\theta}{\cos^2\theta+\sin^2\theta}$ $=\dfrac{\cos^2\theta}{1}$ $=\cos^2\theta$ $=\text{RHS}$ **OR/*OF***	✓ writing as a single fraction/*skryf as enkelbreuk* ✓ quotient identity / *kwosiëntidentiteit* ✓ $\times\dfrac{\cos^2\theta}{\cos^2\theta}$ ✓ square identity/*vierkantidentiteit* (4) ✓ quotient identity/

4.2	$\text{LHS}=1-\left(\dfrac{\sin^2\theta}{\cos^2\theta}\div\left(1+\dfrac{\sin^2\theta}{\cos^2\theta}\right)\right)$ $=1-\left(\dfrac{\sin^2\theta}{\cos^2\theta}\times\dfrac{\cos^2\theta}{\cos^2\theta+\sin^2\theta}\right)$ $=1-\left(\dfrac{\sin^2\theta}{\cos^2\theta}\times\dfrac{\cos^2\theta}{1}\right)$ $=1-\sin^2\theta$ $=\cos^2\theta$ $=\text{RHS}$	*kwosiëntidentiteit* ✓ writing as a single fraction/ *skryf as enkelbreuk* ✓ square identity/*vierkantidentiteit* ✓ simplification/*vereenvoudiging* (4)

4.3	$\cos^2\frac{1}{2}x=\frac{1}{4}$ $\cos\frac{1}{2}x=\frac{1}{2}\ or\ -\frac{1}{2}$ $\frac{1}{2}x=60°+k.360°$ or $\frac{1}{2}x=300°+k.360°$ or $\frac{1}{2}x=120°+k.360°$ or $\frac{1}{2}x=240°+k.360°$ $x=120°+k.720°$ or $x=600°+k.720°$ or $x=240°+k.720°$ or $x=480°+k.720°$; $k\in Z$	✓✓ $\cos^2\frac{1}{2}x=\frac{1}{4}$ ✓ 60° and 300° ✓ 120° and 240° ✓ write at least one general solution as $\frac{1}{2}x=\angle+k.360°$ ✓ write at least one general solution as $x=\angle+k.720°$; $k\in Z$ (6)
	OR/*OF*	
	$\cos^2\frac{1}{2}x=\frac{1}{4}$ $\cos\frac{1}{2}x=\frac{1}{2}\ or\ -\frac{1}{2}$ $\frac{1}{2}x=\pm60°+k.360°$ or $\frac{1}{2}x=\pm120°+k.360°$ $x=\pm120°+k.720°$ or $x=\pm240°+k.720°$; $k\in Z$	✓✓ $\cos^2\frac{1}{2}x=\frac{1}{4}$ ✓ ±60° ✓±120° ✓ write at least one general solution as $\frac{1}{2}x=\angle+k.360°$ ✓ write at least one general solution as $x=\angle+k.720°$ $k\in Z$ (6)

4.4.1	$\sin(A-B)=\cos[90°-(A-B)]$ $=\cos[(90°-A)-(-B)]$ $=\cos(90°-A)\cos(-B)+\sin(90°-A)\sin(-B)$ $=\sin A\cos B+\cos A(-\sin B)$ $=\sin A\cos B-\cos A\sin B$	✓ co–ratio/*ko-verhouding* ✓ writing as a difference of A & B/ *skryf as verskil van A & B* ✓ expansion/*uitbreiding* ✓ all reductions/*alle reduksies* (4)
	OR/*OF*	
	$\sin(A-B)=\cos[90°-(A-B)]$ $=\cos[(90°+B)-A]$ $=\cos(90°+B)\cos A+\sin(90°+B)\sin A$ $=-\sin B\cos A+\cos B\sin A$ $=\sin A\cos B-\cos A\sin B$	✓ co–ratio/*ko-verhouding* ✓ writing as a difference of A & B/ *skryf as verskil van A & B* ✓ expansion/*uitbreiding* ✓ all reductions/*alle reduksies* (4)
4.4.2	$\sin(x+64°)\cos(x+379°)+\sin(x+19°)\cos(x+244°)$ $=\sin(x+64°)\cos(x+19°)+\sin(x+19°)[-\cos(x+64°)]$ $=\sin(x+64°)\cos(x+19°)-\cos(x+64°)\sin(x+19°)$ $=\sin[x+64°-(x+19°)]$ $=\sin 45°$ $=\frac{1}{\sqrt{2}}$	✓ $\cos(x+379°)=\cos(x+19°)$ ✓✓ $\cos(x+244°)=-\cos(x+64°)$ ✓✓compound formula identity/ *saamgestelde identiteit* ✓ $\sin 45°$ (6) **[23]**

QUESTION/*VRAAG* 5

5.1.1	$\cos 2\theta = -\frac{5}{6}$, where $2\theta \in [180°; 270°]$	
		✓ diagram (3rd quadrant only)
	$y^2 = 6^2 - (-5)^2$ [Pythagoras]	✓ using Pythagoras
	$y = \pm\sqrt{11}$	
	(5 ; y) is in 3rd quadrant:	
	$\therefore y = -\sqrt{11}$	✓ y – value
no calculator in 5.1	$\sin 2\theta = -\frac{\sqrt{11}}{6}$	✓ answer (4)
	Getting to $\sin 2\theta = \frac{\sqrt{11}}{6}$: 3/4	
	OR/*OF*	
	$\sin^2 2\theta = 1 - \cos^2 2\theta$	✓ $\sin^2 2\theta = 1 - \cos^2 2\theta$
	$= 1 - \left(-\frac{5}{6}\right)^2$	✓ substitution
	$= 1 - \frac{25}{36}$	
	$= \frac{11}{36}$	✓ value of $\sin^2 2\theta$
	$\sin 2\theta = -\frac{\sqrt{11}}{6}$	✓ answer (4)

5.1.2	$\cos 2\theta = 1 - 2\sin^2\theta$ $2\sin^2\theta = 1 - \cos 2\theta$ $\sin^2\theta = \dfrac{1-\left(-\frac{5}{6}\right)}{2}$ $= \dfrac{11}{6} \times \dfrac{1}{2}$ $= \dfrac{11}{12}$	✓ $\cos 2\theta = 1 - 2\sin^2\theta$ ✓ substitution ✓ answer (3)

5.2	$\sin(180° - x).\cos(-x) + \cos(90° + x).\cos(x - 180°)$ $= \sin x.\cos x - \sin x(-\cos x)$ $= 2\sin x.\cos x$ $= \sin 2x$	✓ $\sin x$ ✓ $\cos x$ ✓ $-\sin x$ ✓ $\cos x$ ✓ simplification ✓ answer (6)
5.3	$\sin 3x.\cos y + \cos 3x.\sin y$ $\sin(3x + y)$ $= \sin 270°$ $= -1$	✓ compound angle ✓ answer (2)
5.4.1	$2\cos x = 3\tan x$ $2\cos x = \dfrac{3\sin x}{\cos x}$ $2\cos^2 x = 3\sin x$ $2(1 - \sin^2 x) = 3\sin x$ $2 - 2\sin^2 x = 3\sin x$ $2\sin^2 x + 3\sin x - 2 = 0$	✓ $\tan x = \dfrac{\sin x}{\cos x}$ ✓ multiplying by $\cos\theta$ ✓ $\cos^2 x = 1 - \sin^2 x$ (3)

5.4.2	$2\sin^2 x + 3\sin x - 2 = 0$ $(2\sin x - 1)(\sin x + 2) = 0$ $\sin x = \frac{1}{2}$ or $\sin x = -2$ (no solution) $x = 30° + k.360°$ or $x = 150° + k.360°$; $k \in Z$	✓ factors ✓ both values of $\sin x$ ✓ no solution ✓ $30° + k.360°$ ✓ $150° + k.360°$; $k \in Z$ (5)
5.4.3	$5y = 30° + k.360°$ or $5y = 150° + k.360°$ $y = 6° + k.72°$ or $y = 30° + k.72°$ $\therefore y = 144° + 6°$ or $y = 144° + 30°$ $y = 150°$ or $y = 174°$ **OR/*OF*** $144° \leq y \leq 216°$ $720° \leq 5y \leq 1080°$ $5y = 750°$ or $5y = 870°$ $y = 150°$ or $y = 174°$	✓ $y = 6° + k.72°$ ✓ $y = 30° + k.72°$ ✓ $150°$ ✓ $174°$ (4) ✓ $5y = 750°$ ✓ $5y = 870°$ ✓ $150°$ ✓ $174°$ (4)

5.5.1	$g(x) = -4\cos(x + 30°)$ maximum value = 4	✓ answer (1)
5.5.2	range of/*waardeversameling van* $g(x)$: $-4 \leq y \leq 4$ **OR/*OF*** $y \in [-4 ; 4]$ $\therefore$ range of/*waardeversameling van* $g(x) + 1$: $-3 \leq y \leq 5$ **OR/*OF*** $y \in [-3 ; 5]$	✓ range of $g(x)$ ✓ answer (2)

5.5.3	$y = -4\cos(x + 30°)$ shifted to the left/*skuif na links*: $y = -4\cos(x + 30° + 60°)$ $= -4\cos(x + 90°)$ $= 4\sin x$ $\therefore h(x) = -4\sin x$	✓ shift of 60° to the left ✓ reduction ✓ equation of h (3) **[33]**

WORKSHEETS, QUESTION 6-11

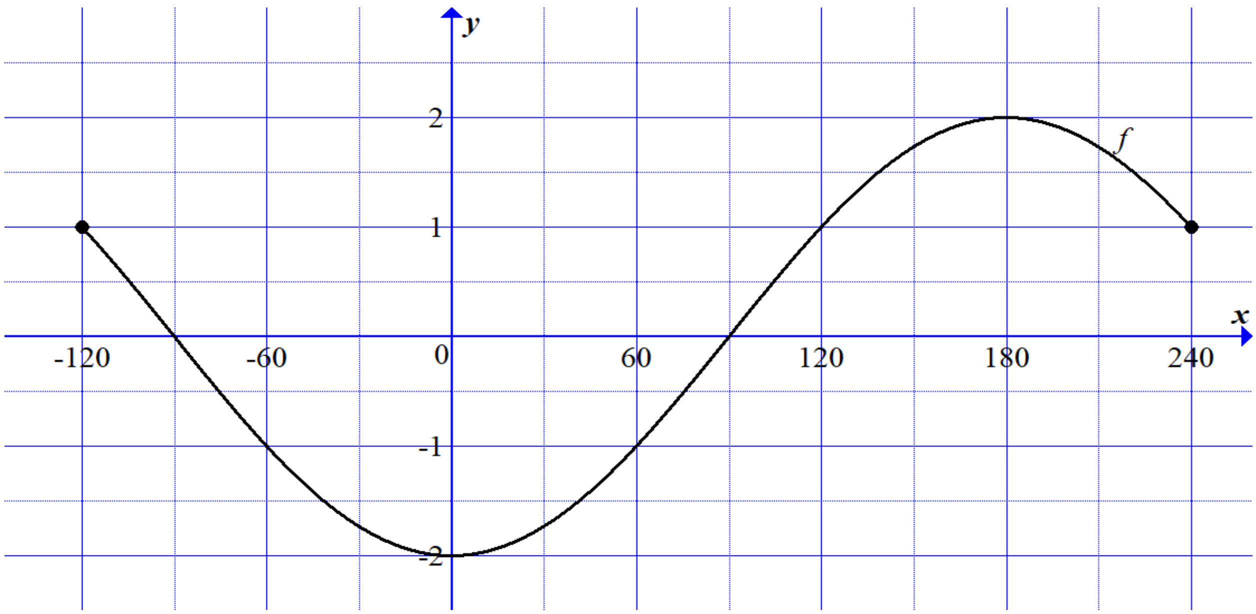
QUESTION 6

Given the equation: $\sin(x + 60°) + 2\cos x = 0$

6.1 Show that the equation can be rewritten as $\tan x = -4 - \sqrt{3}$. (4)

6.2 Determine the solutions of the equation $\sin(x + 60°) + 2\cos x = 0$ in the interval $-180° \leq x \leq 180°$. (3)

6.3 In the diagram below, the graph of $f(x) = -2\cos x$ is drawn for $-120° \leq x \leq 240°$.



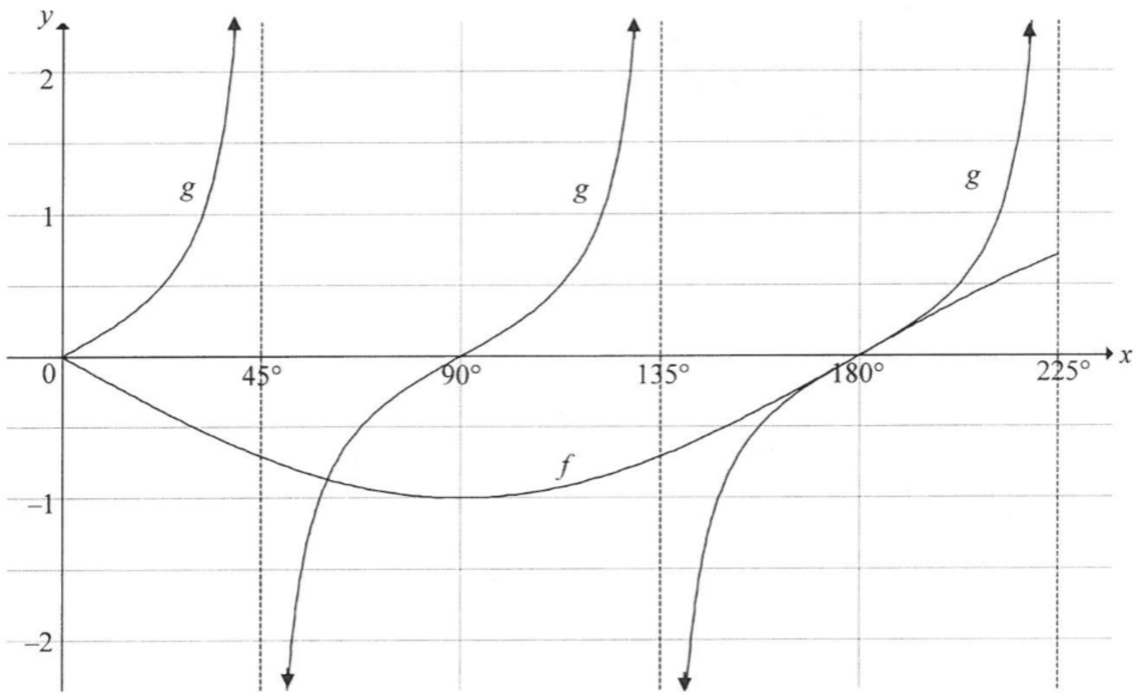
6.3.1 Draw the graph of $g(x) = \sin(x + 60°)$ for $-120° \leq x \leq 240°$ on the grid provided in the ANSWER BOOK. (3)

6.3.2 Determine the values of x in the interval $-120° \leq x \leq 240°$ for which $\sin(x + 60°) + 2\cos x > 0$. (3)

[13]

QUESTION 7

In the diagram, the graphs of the functions $f(x) = a\sin x$ and $g(x) = \tan bx$ are drawn on the same system of axes for the interval $0° \leq x \leq 225°$.

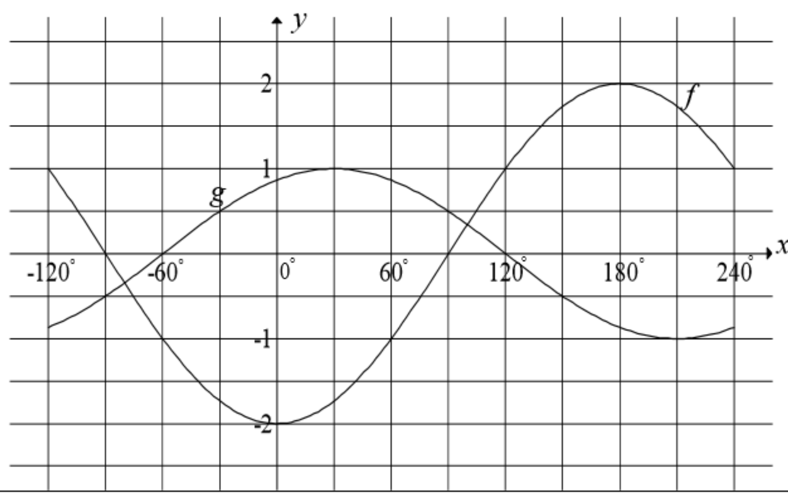


7.1 Write down the values of a and b. (2)

7.2 Write down the period of $f(3x)$. (2)

7.3 Determine the values of x in the interval $90° \leq x \leq 225°$ for which $f(x).g(x) \leq 0$. (3)
[7]

QUESTION/*VRAAG* 6

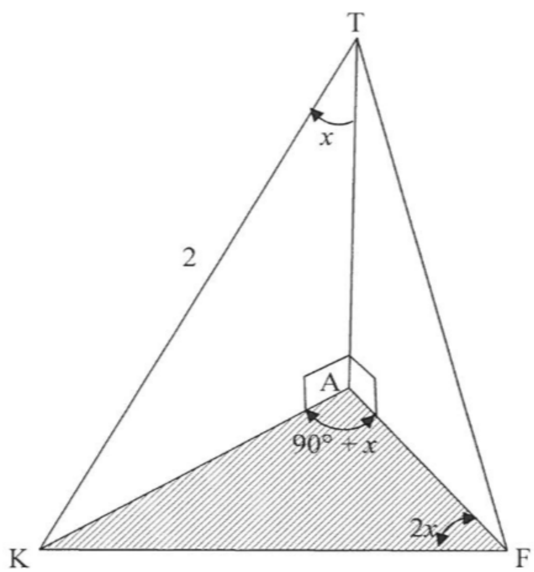
6.1	$\sin(x + 60°) + 2\cos x = 0$ $\sin x \cos 60° + \cos x \sin 60° + 2\cos x = 0$ $\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x + 2\cos x = 0$ $\frac{1}{2}\sin x = -2\cos x - \frac{\sqrt{3}}{2}\cos x$ $\sin x = -4\cos x - \sqrt{3}\cos x$ $\sin x = \cos x(-4 - \sqrt{3})$ $\frac{\sin x}{\cos x} = \frac{\cos x(-4-\sqrt{3})}{\cos x}$ $\therefore \tan x = -4 - \sqrt{3}$	✓expansion/*uitbreiding* ✓special angle values/ *spesiale ∠-waardes* ✓simpl/*vereenv* ✓ $\sin x = \cos x(-4 - \sqrt{3})$ (4)
6.2	$\tan x = -4 - \sqrt{3}$ $\tan x = -(4 + \sqrt{3})$ *ref* ∠ = 80,10° $x = -80{,}1°$ or/*of* 99,9°	✓80,10° ✓99,90° ✓–80,1° (3)
6.3.1	 	✓(30° ; 1) ✓(–60° ; 0) ✓shape/*vorm* (3)
6.3.2	$\therefore \sin(x + 60°) > -2\cos x$ $x \in (-80{,}10° ; 99{,}90°)$ **OR/*OF*** $-80{,}10° < x < 99{,}90°$	✓✓critical values/ *kritiese waardes* ✓notation/*notasie* (3) **[13]**

QUESTION7

7.1	$a=-1$ $b=2$	✓ answer ✓ answer (2)
7.2	$f(3x) = -\sin 3x$ Period of $f(3x) = \frac{360°}{3}$ $= 120°$ Answer only: Full marks	✓ $\frac{360°}{3}$ ✓ answer (2)
7.3	$x \in [90°\ ;\ 135°) \cup \{180°\}$ **OR/*OF*** $90° \leq x < 135°$ or $x = 180°$	✓ 90° and 135° in interval form ✓ 180° as single value ✓ correct brackets (3) ✓ 90° and 135° in interval form ✓ 180° as single value ✓ correct inequalities (3) **[7]**

QUESTION 8

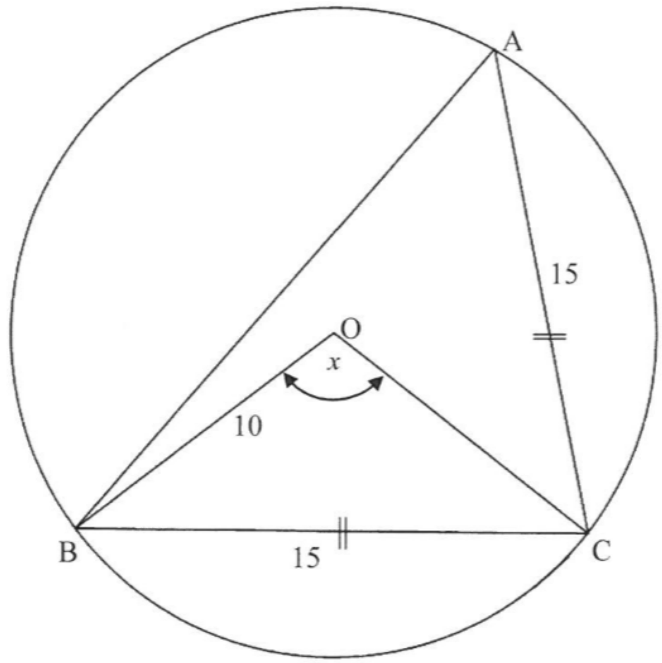
8.1 In the figure, points K, A and F lie in the same horizontal plane and TA represents a vertical tower. $A\hat{T}K = x$, $K\hat{A}F = 90° + x$ and $K\hat{F}A = 2x$ where $0° < x < 30°$. $TK = 2$ units.



8.1.1 Express AK in terms of $\sin x$. (2)

8.1.2 Calculate the numerical value of KF. (5)

8.2 In the diagram below, a circle with centre O passes through A, B and C. BC = AC = 15 units. BO and OC are joined. OB = 10 units and $B\hat{O}C = x$.



Calculate:

8.2.1 The size of x (4)

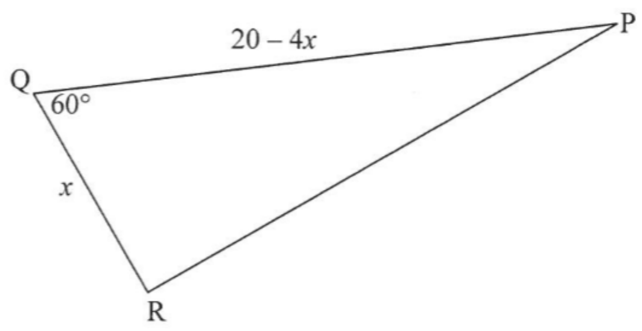
8.2.2 The size of $A\hat{C}B$ (3)

8.2.3 The area of ΔABC (2)

[16]

QUESTION 9

9.1 In the diagram below, ΔPQR is drawn with $PQ = 20 - 4x$, $RQ = x$ and $\hat{Q} = 60°$.

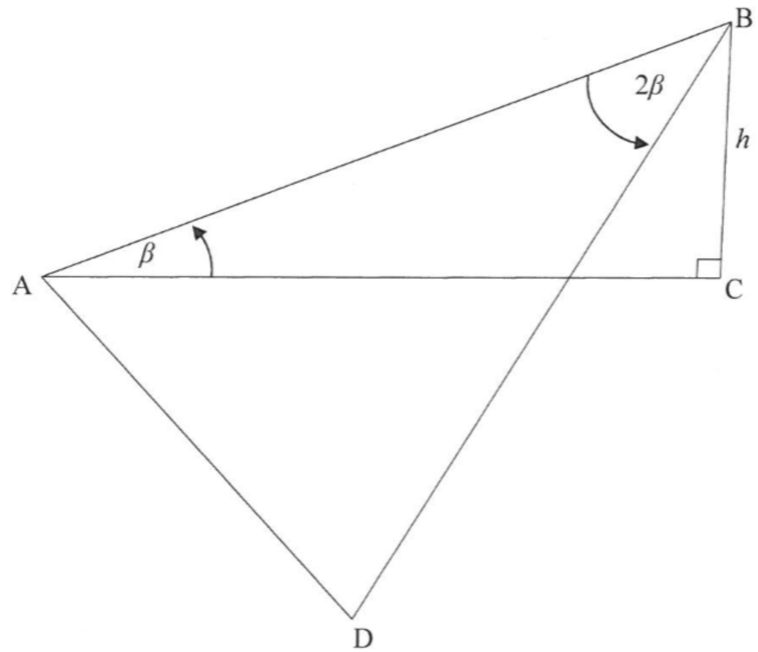


9.1.1 Show that the area of $\Delta PQR = 5\sqrt{3}x - \sqrt{3}x^2$. (2)

9.1.2 Determine the value of x for which the area of ΔPQR will be a maximum. (3)

9.1.3 Calculate the length of PR if the area of ΔPQR is a maximum. (3)

9.2 In the diagram below, BC is a pole anchored by two cables at A and D. A, D and C are in the same horizontal plane. The height of the pole is h and the angle of elevation from A to the top of the pole, B, is β. $A\hat{B}D = 2\beta$ and $BA = BD$.

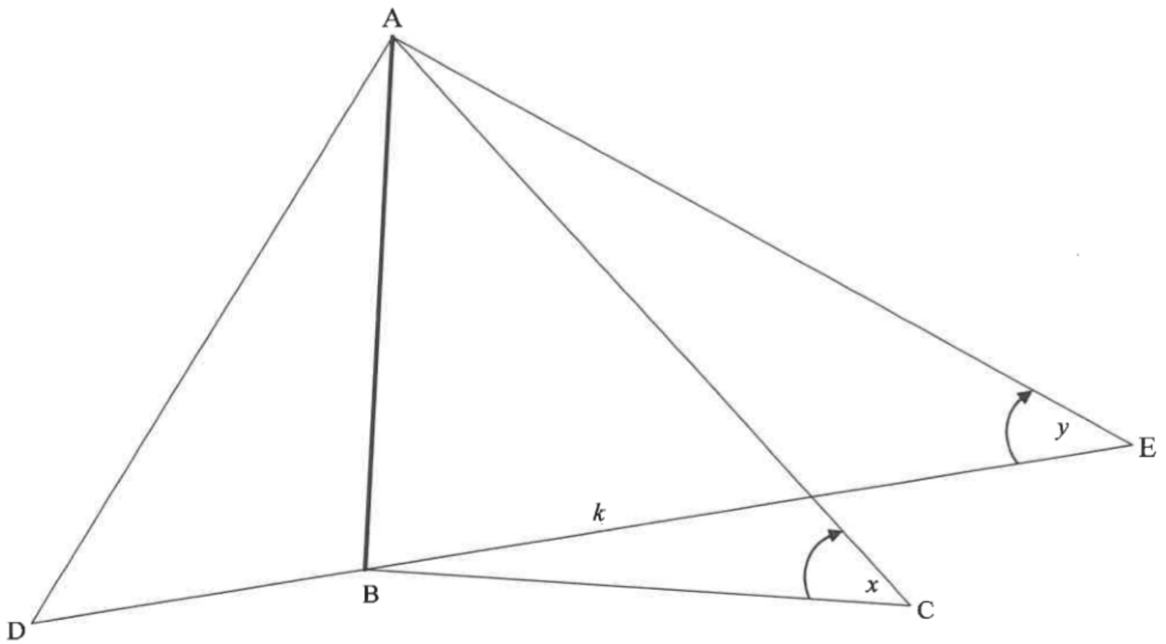


Determine the distance AD between the two anchors in terms of h. (7)

[15]

QUESTION 10

AB represents a vertical netball pole. Two players are positioned on either side of the netball pole at points D and E such that D, B and E are on the same straight line. A third player is positioned at C. The points B, C, D and E are in the same horizontal plane. The angles of elevation from C to A and from E to A are x and y respectively. The distance from B to E is k.



10.1 Write down the size of $A\hat{B}C$. (1)

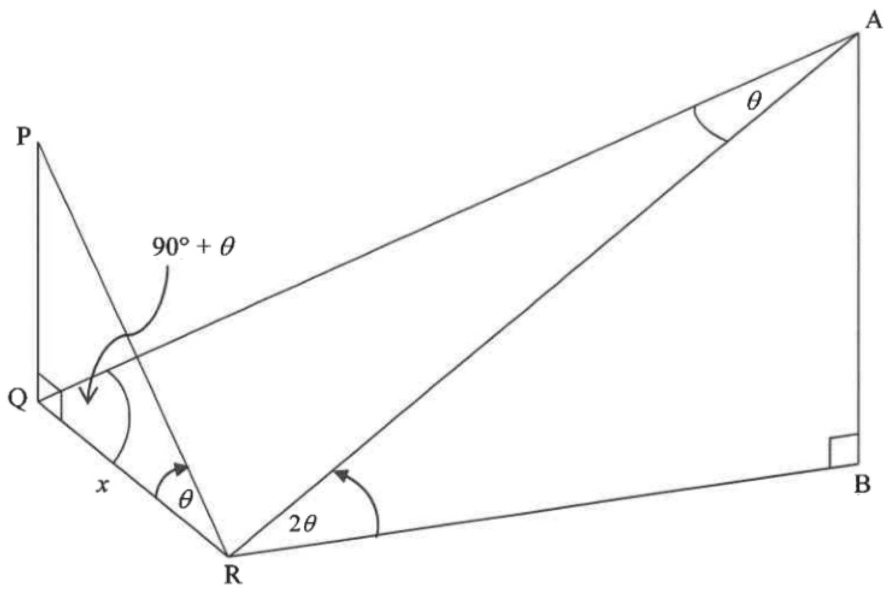
10.2 Show that $AC = \dfrac{k.\tan y}{\sin x}$ (4)

10.3 If it is further given that $D\hat{A}C = 2x$ and AD = AC, show that the distance DC between the players at D and C is $2k \tan y$. (5)

[10]

QUESTION 11

PQ and AB are two vertical towers.
From a point R in the same horizontal plane as Q and B, the angles of elevation to P and A are θ and 2θ respectively.
$A\hat{Q}R = 90° + \theta$, $Q\hat{A}R = \theta$ and $QR = x$.



11.1 Determine in terms of x and θ:

6.1.1 QP (2)

6.1.2 AR (2)

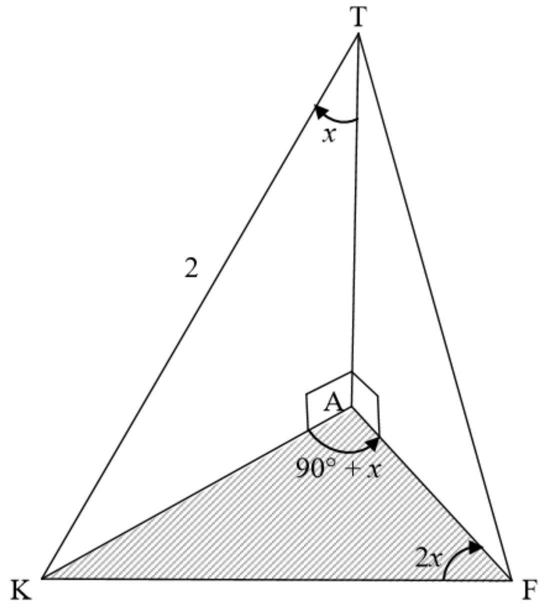
11.2 Show that $AB = 2x\cos^2\theta$ (4)

11.3 Determine $\frac{AB}{QP}$ if $\theta = 12°$. (2)

[10]

QUESTION 8

8.1

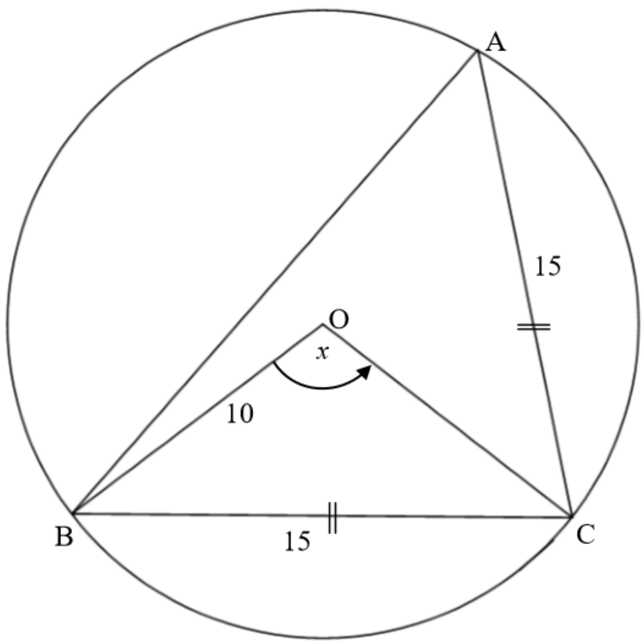


8.1.1	In ΔTAK: $\frac{AK}{KT} = \sin K\hat{T}A$ $AK = KT.\sin x$ $= 2\sin x$ **OR/*OF*** $\frac{\sin K\hat{T}A}{AK} = \frac{\sin K\hat{A}T}{KT}$ $\frac{\sin 90^\circ}{2} = \frac{\sin x}{AK}$ $AK = 2\sin x$	✓ correct trig ratio/ *korrekte trigverh.* ✓answer/*antw* (2) ✓ correct subst into sine rule/*korrekte subst in sin-reël* ✓answer/*antw* (2)

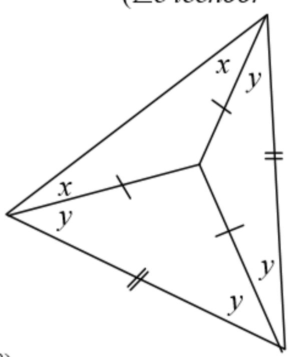
8.1.2		
	In ΔAKF: $\frac{KF}{\sin K\hat{A}F}=\frac{AK}{\sin A\hat{F}K}$ $\frac{KF}{\sin(90°+x)}=\frac{AK}{\sin 2x}$ $KF=\frac{AK.\sin(90°+x)}{\sin 2x}$ $=\frac{2\sin x.\cos x}{2\sin x.\cos x}$ $=1$ **OR/*OF***	✓ using sine rule/ *gebruik sin-reël* ✓ correct subst into sine rule/*korrekte subst in sin-reël* ✓ $\sin(90°+x)=\cos x$ ✓ $2\sin x.\cos x$ ✓ 1 (5)

8.1.2			
	In ΔAKF: $\frac{KF}{\sin K\hat{A}F}=\frac{AK}{\sin A\hat{F}K}$ $\frac{KF}{\sin(90°+x)}=\frac{AK}{\sin 2x}$ $KF=\frac{AK.\sin(90°+x)}{\sin 2x}$ $=\frac{AT.\tan x.\cos x}{2\sin x.\cos x}$ $=\frac{2\cos x.\frac{\sin x}{\cos x}.\cos x}{2\sin x.\cos x}$ $=1$	$\cos x=\frac{AT}{2}$ $\therefore AT=2\cos x$	✓ using sine rule/ *gebruik sin-reël* ✓ correct subst into sine rule/*korrekte subst in sin-reël* ✓ $\sin(90°+x)=\cos x$ ✓ $2\sin x.\cos x$ ✓ 1 (5)

8.2



8.2.1	In ΔBOC: $BC^2 = BO^2 + CO^2 - 2.BO.\ CO\ .\cos x$ $15^2 = 10^2 + 10^2 - 2(10)(10).\ \cos x$ $200 \cos x = -25$ $\cos x = -0{,}125$ $x = 180° - 82{,}82°$ $= 97{,}18°$	✓ using cosine rule/ *gebruik cos-reël* ✓correct subst/ *korrekte subst* ✓ $\cos x = -0{,}125$ ✓97,18° (4)
	OR/*OF*	
	Draw a line OD⊥ BC: BD = DC (line from centre ⊥ on chord) ΔOBD ≡ ΔOCD (90°; h; s) $\sin \frac{x}{2} = \frac{7{,}5}{10}$ $\frac{x}{2} = 48{,}59°$ $\therefore\ x = 97{,}18°$	✓ S/R ✓correct ratio/ *korrekte verh* ✓ value of/*waarde van* $\frac{x}{2}$ ✓97,18° (4)

8.2.2	$B\hat{A}C = 48,59°$ ($\angle$ at centre=2×$\angle$at circ/$\angle$ *by midpt=2×$\angle$omt*) $A\hat{B}C = B\hat{A}C = 48,59°$ ($\angle$'s opp equal sides/$\angle$*e teenoor = sye*) $\therefore A\hat{C}B = 82,82°$ (sum of $\angle$s of Δ/*som van $\angle$e van Δ*)	✓ S ✓ S ✓82,82° (3)
	OR/*OF* $A\hat{C}B = \frac{1}{2} A\hat{O}B$ ($\angle$at centre = 2× $\angle$at circle) ($\angle$ *by midpt=2×$\angle$omt*) $= \frac{1}{2}[360° - 2(97,18°)]$ $= 82,82°$	✓ S ✓ S ✓82,82° (3)
	OR/*OF* $O\hat{C}B = \frac{1}{2}(180° - 97,18°)$ ($\angle$'s opp equal sides; sum of $\angle$s of Δ) $= 41,41°$ ($\angle$*e teenoor = sye; som van $\angle$e van Δ*)  $A\hat{C}B = 2(41,41°)$ $= 82,82°$	✓ S ✓ S ✓82,82° (3)
8.2.3	Area/*Oppervlakte* ΔABC $= \frac{1}{2}(BC)(AC)\sin A\hat{C}B$ $= \frac{1}{2}(15)(15)(\sin 82,82°)$ $= 111,62 \text{ cm}^2$	✓correct subst into area rule/*korrekte subst in opp-reël* ✓ $111,62 \text{ cm}^2$ (2) **[16]**

QESTION 9

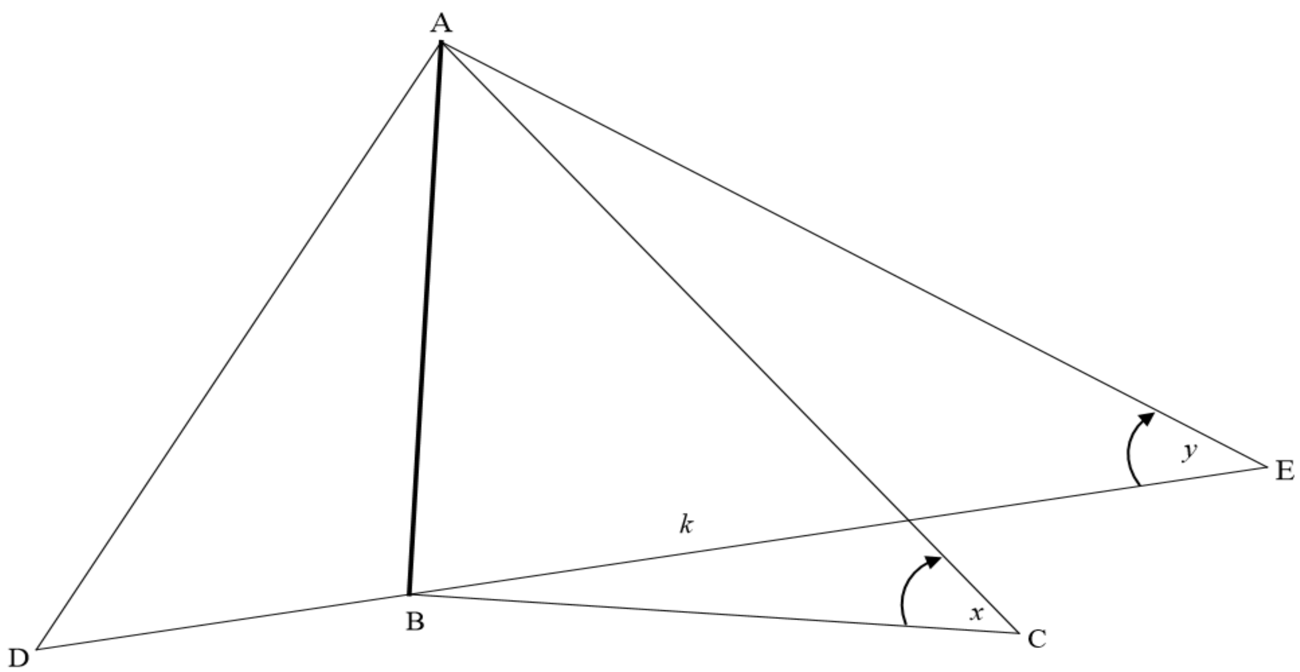
9.1.1	Area of/*Oppervlakte van* $\Delta PQR = \frac{1}{2} PQ.QR.\sin\hat{Q}$ **1.1.1** $= \frac{1}{2}x(20-4x)(\sin 60°)$ **1.1.2** $= 10x - 2x^2\left(\frac{\sqrt{3}}{2}\right)$ **1.1.** $= 5\sqrt{3}x - \sqrt{3}x^2$	✓subst into area rule/ *subst in opp-reël* ✓subst & simpl/ *subst en vereenv* (2)
9.1.2	For maximum area/*Vir maksimum opp:* $(\text{Area } \Delta PQR)' = 0$ $5\sqrt{3} - 2\sqrt{3}x = 0$ $2\sqrt{3}x = 5\sqrt{3}$ $\therefore\ x_{max} = \frac{5}{2}$ or $2\frac{1}{2}$ or/*of* 2,5	✓$(\text{Area } \Delta PQR)' = 0$ ✓$5\sqrt{3} - 2\sqrt{3}x$ ✓answ/*antw* (3)
	OR/*OF* $x_{max} = -\frac{b}{2a}$ $= -\frac{5\sqrt{3}}{2(-\sqrt{3})} = \frac{5}{2}$ or $2\frac{1}{2}$ or 2,5	✓formula/*e* ✓ subst ✓answ/*antw* (3)
	OR/*OF*	

9.1.2	$5\sqrt{3}x-\sqrt{3}x^2=0$ $\sqrt{3}x(5-x)=0$ $\therefore\ x=0$ or 5 $\therefore\ x_{max}=\frac{0+5}{2}=\frac{5}{2}$ or/*of* 2,5	✓x-intercepts/ *x-afsnitte* ✓subst ✓answ/*antw* (3)
9.1.2	$RP^2=QP^2+QR^2-2.QP.QR.\cos Q$ $=10^2+2{,}5^2-2(10)(2{,}5)\cos 60°$ $=81{,}25$ $\therefore\ RP=9{,}01$	✓subst into cosine rule/*in cos-reël* ✓simpl/*vereenv* ✓answ/*antw* (3)

9.2	In ΔABC: $\sin\beta=\frac{h}{AB}$ $\therefore\ AB=\frac{h}{\sin\beta}$ In ΔABD: AB = BD and/*en* $A\hat{D}B=90°-\beta$ [$\angle$s of/*v* $\Delta=180°$] $\frac{\sin 2\beta}{AD}=\frac{\sin(90°-\beta)}{AB}$ $AD=\frac{AB.\sin 2\beta}{\sin(90°-\beta)}$ $=\frac{h}{\sin\beta}\times\frac{2\sin\beta.\cos\beta}{\cos\beta}$ $=2h$ **OR/*OF***	✓AB ito h and/*en* β ✓ $A\hat{D}B=90°-\beta$ ✓correct subst into cosine rule/*subst korrek in cos-reël* ✓AD as subject/ *onderwerp* ✓expansion/*uitbrei* ✓sin (90°– β) = cos β ✓answer ito h (7)

9.2	In ΔABC: $\sin\beta = \frac{h}{\text{AB}}$ $\therefore \text{AB} = \frac{h}{\sin\beta}$ In ΔABD: AB = BD $\text{AD}^2 = \text{AB}^2 + \text{AB}^2 - 2\text{AB.AB}.\cos 2\beta$ $= \left(\frac{h}{\sin\beta}\right)^2 + \left(\frac{h}{\sin\beta}\right)^2 - 2\left(\frac{h}{\sin\beta}\right)^2 .\cos 2\beta$ $= \left(\frac{h}{\sin\beta}\right)^2 + \left(\frac{h}{\sin\beta}\right)^2 - 2\left(\frac{h}{\sin\beta}\right)^2 (1 - 2\sin^2\beta)$ $= \left(\frac{h}{\sin\beta}\right)^2 + \left(\frac{h}{\sin\beta}\right)^2 - 2\left(\frac{h}{\sin\beta}\right)^2 + 4h^2$ $= 4h^2$ $\therefore \text{AD} = 2h$ **OR/*OF*** Split isosceles triangle ABQ into two congruent triangles AEB and DEB. Then $\Delta\text{ABC} \equiv \Delta\text{BAE}$ (AB = AC, $\text{A}\hat{\text{B}}\text{E} = \text{B}\hat{\text{A}}\text{C} = \beta$, h) $\therefore$ AE = ED = BC = h $\therefore \text{AD} = 2h$	✓AB ito h and/*en* β ✓correct subst into cosine rule/*subst korrek in cos-reël* ✓expansion/*uitbrei* ✓multiplication/*vermenigv* ✓simpl/*vereenv* ✓answer ito h (7) (7)
		[15]

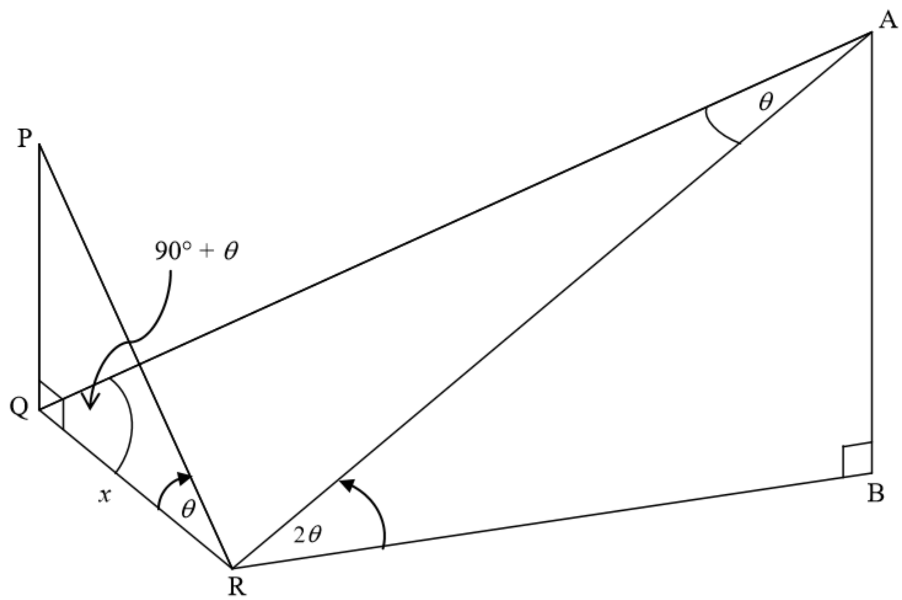
QUESTION 10



10.1	$A\hat{B}C = 90°$	✓ answer (1)
10.2	In Δ ABE: $\frac{AB}{BE} = \tan y$ $AB = k \tan y$ In Δ ABC: $\frac{AB}{AC} = \sin x$ $AC = \frac{AB}{\sin x}$ $= \frac{k \tan y}{\sin x}$	✓ correct ratio ✓ value AB ✓ correct ratio ✓ AC as subject and substitution (4)

10.3	$\hat{ADC} = \hat{ACD} = \frac{180° - 2x}{2} = 90° - x$ $\frac{DC}{\sin 2x} = \frac{AC}{\sin(90° - x)}$ $\frac{DC}{2\sin x \cos x} = \frac{AC}{\cos x}$ $DC = \frac{AC(2\sin x \cos x)}{\cos x}$ $= \frac{k\tan y}{\sin x} \cdot \frac{2\sin x\cos x}{\cos x}$ $= 2k\tan y$	✓ $90° - x$ ✓subst into sine rule ✓ $2\sin x\cos x$ ✓ $\cos x$ ✓ substitution (5)
	OR/*OF* $DC^2 = AD^2 + AC^2 - 2AD.AC\cos 2x$ $= AC^2 + AC^2 - 2AC^2\cos 2x$ $= 2AC^2(1 - \cos 2x)$ $= 2AC^2(1 - 1 + \sin^2 x)$ $= 4AC^2\sin^2 x$ $DC = 2AC.\sin x$ $= 2\left(\frac{k.\tan y}{\sin x}\right).\sin x$ $= 2k.\tan y$	✓substitution into cos rule ✓ factorisation ✓ $1 - 2\sin^2 x$ ✓ DC ito AC and $\sin x$ ✓ substitution (5)
	OR/*OF* $DC^2 = AD^2 + AC^2 - 2AD.AC\cos 2x$ $= 2\left(\frac{k\tan y}{\sin x}\right)^2 - 2\left(\frac{k\tan y}{\sin x}\right)^2 \cos 2x$ $= \frac{2k^2\tan^2 y}{\sin^2 x} - \frac{2k^2\tan^2 y}{\sin^2 x}(1 - 2\sin^2 x)$ $= \frac{2k^2\tan^2 y}{\sin^2 x} - \frac{2k^2\tan^2 y}{\sin^2 x} + 4k^2\tan^2 y$ $DC = \sqrt{4k^2\tan^2 y}$ $= 2k\tan y$	✓correct cos rule ✓ substitution ✓ $1 - 2\sin^2 x$ ✓ squaring and multiplication ✓ $\sqrt{4k^2\tan^2 y}$ (5) **[10]**

QUESTION 11



11.1.1	$\tan\theta = \frac{PQ}{QR} = \frac{PQ}{x}$ $\therefore PQ = x\tan\theta$ Answer only: full marks **OR/*OF*** $\frac{QR}{\sin P} = \frac{PQ}{\sin P\hat{R}Q}$ $\therefore PQ = \frac{x.\sin\theta}{\sin(90° - \theta)}$	✓ trig ratio ✓ answer (2) ✓ trig ratio ✓ answer (2)
11.1.2	$\frac{AR}{\sin A\hat{Q}R} = \frac{QR}{\sin Q\hat{A}R}$ $AR = \frac{x\sin(90° + \theta)}{\sin\theta}$ Answer only: full marks	✓ use of sine rule ✓ substitution into sine rule correctly (2)

11.2	$\sin 2\theta = \frac{AB}{AR}$ $AB = AR\sin 2\theta$ $= \frac{x\sin(90° + \theta).\sin 2\theta}{\sin\theta}$ $= \frac{x\cos\theta.\sin 2\theta}{\sin\theta}$ $= \frac{x\cos\theta.2\sin\theta\cos\theta}{\sin\theta}$ $= 2x\cos^2\theta$	✓ substitution into trig ratio and AB as subject ✓ substitution of AR ✓ co-ratio ✓ $\sin 2\theta = 2\sin\theta\cos\theta$ (4)
11.3	$\frac{AB}{QP} = \frac{2x\cos^2 12°}{x\tan 12°}$ $= 9$	✓ substitution CA from 6.1.1) ✓ answer (2) **[10]**

2. EUCLIDEAN GEOMETRY

2.1 WORK COVERED

- All theorems on straight lines, triangles and parallel lines
- Theorem of Pythagoras
- Similarity and Congruency
- Midpoint theorem
- Properties of quadrilaterals
- Circle Geometry.
- Proportionality theorems
- Similar triangles
- Theorem of Pythagoras (proof by similar triangles)

2.2 OVERVIEW OF TOPICS

GRADE 10

- Revise basic results established in earlier grades regarding lines, angles and triangles, especially the similarity and congruence of triangles.
- Investigate line segments joining the midpoints of two sides of a triangle.
- Define the following special quadrilaterals: the kite, parallelogram, rectangle, rhombus, square and trapezium. Investigate and make conjectures about the properties of the sides, angles, diagonals and areas.

GRADE 11

- Investigate and prove theorems of the geometry of circles assuming results from earlier grades, together with one other result concerning tangents and radii of circles.
- Solve circle geometry problems, providing reasons for statements when required.
- Prove riders.

GRADE 12

- Revise earlier (Grade 9-11) work on similar polygons.
- Prove (accepting results established in earlier grades):
  - that a line drawn parallel to one side of a triangle divides the other two sides proportionally (and the Mid-point theorem as a special case of this theorem);
  - that equiangular triangles are similar;
  - that triangles with sides in proportion are similar;
  - the Pythagorean Theorem by similar triangles and riders

	According to the National Diagnostic Reports, the previous learners had challenges to:
	• Make assumptions on cyclic quadrilaterals as equal, right angles where there is none, angles as equal, lines as parallel, etc. • When proving cyclic quad or that a line is a tangent they use that as a reason in their proof. • State incomplete or incorrect reasons for statements • Identify correct sides that are in proportion • State proportions without reasons • Write proof of a theorem without making the necessary construction • Differentiate when to use similarity or congruency when solving riders • Understand properties of quadrilaterals and also the connections between shapes, eg (1) all squares are rectangles (2) all squares are rhombi (3) etc. • Solve problems that integrates topics e.g. Trigonometry and Euclidean Geometry and Analytical Geometry • Prove cyclic quad or // lines or tangents
	SUGGESTIONS TO ADDRESS THE CHALLENGES:
	• Scrutinise the given information and the diagram for clues about which theorems could be used in answering the question. • Differentiate between proving a theorem and applying a theorem • Use the list of reasons provided in the Examination Guidelines. • Identify the correct sides that are in proportion • All statements must be accompanied by reasons. It is essential that the parallel lines be mentioned when stating that corresponding angles are equal, alternate angles are equal, the sum of the co-interior angles is 180° or when stating the proportional intercept theorem. • Note that **construction** is necessary when proving theorems • Understand the difference between the concepts "similarity" and "congruency" • Revise properties of quadrilaterals done in earlier grades • Practise more exercises where the converses of the theorems are used in solving questions • Practice solving problems that integrate topics e.g. Trigonometry and Euclidean Geometry

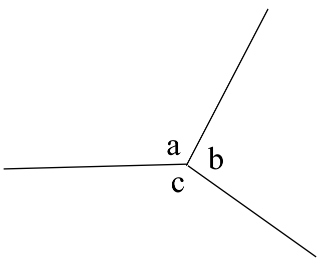
Revision of earlier (Grade 9-10) Geometry

Note:

- You must be able to identify, visualise theorems, axioms to apply in every situation.
- When presented with a diagram they should be able to write the theorem in words.

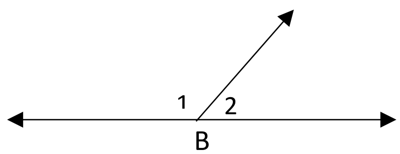
Straight Lines

The sum of angles around a point is 360°



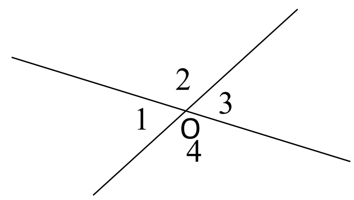
In the diagram, a + b + c = 360°

Adjacent angles on a straight line are supplementary



In the diagram, $\hat{B}_1 + \hat{B}_2 = 180°$

Vertically opposite angles are equal.

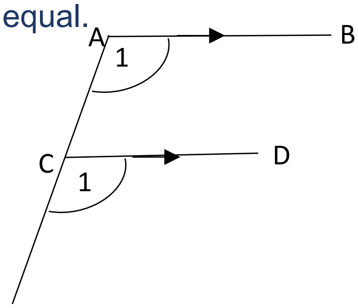


$\hat{O}_1 = \hat{O}_3$ and $\hat{O}_2 = \hat{O}_4$

Parallel Lines

Corresponding angles are equal (F-shape).

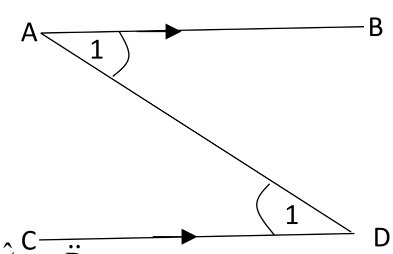
If AB//CD, then the corresponding angles are equal.



$\hat{A}_1 = \hat{C}_1$

Alternate angles are equal (z or N-shape)

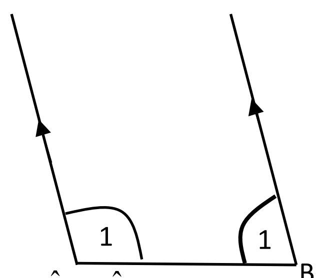
If AB//CD, then alternate angles are equal.



$\hat{A}_1 = \ddot{D}_1$

Co-interior angles are supplementary (U-shape)

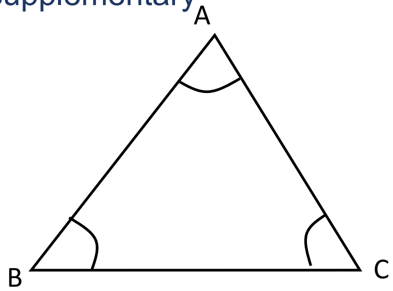
If AB//CD, then co-interior angles are supplementary.



$\hat{A}_1$A+ $\hat{B}_1 = 180°$

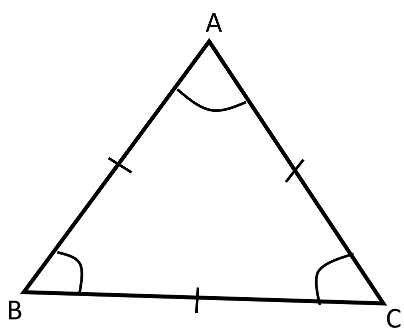
Triangles

The interior angles of a triangle are supplementary



$\hat{A}+\hat{B}+\hat{C}=180°$

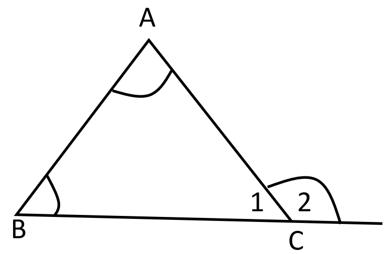
In an equilateral triangle all sides are equal and all angles are equal to $60°$



$\hat{A}=\hat{B}=\hat{C}=60°$, and $AB = BC = AC$

The exterior angle of a triangle is equal to the sum of the interior opposite angles

The interior angles of a triangle are supplementary

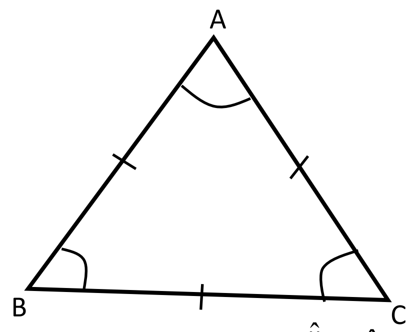


$\hat{C}_2=\hat{A}+\hat{B}$

Angles opposite equal sides are equal

Sides opposite equal angles are equal

In an equilateral triangle all sides are equal and all angles are equal to $60°$



If $AB = AC$, then $\hat{B}=\hat{C}$

$\hat{A}=\hat{B}=\hat{C}=60°$, and $AB = BC = AC$

Conversely, if $\hat{B}=\hat{C}$, then $AB = AC$

The ex

the int

$\hat{A}+\hat{B}$

Angles

Sides

If $AB =$

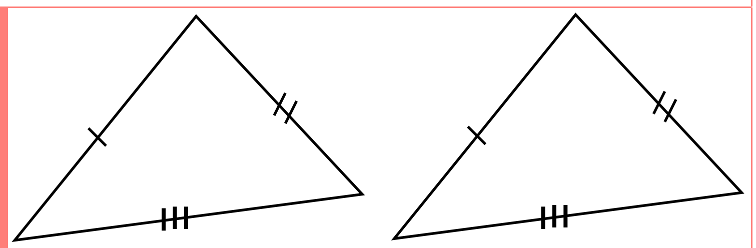
Conve

Congruency

Congruency of triangles (four conditions)

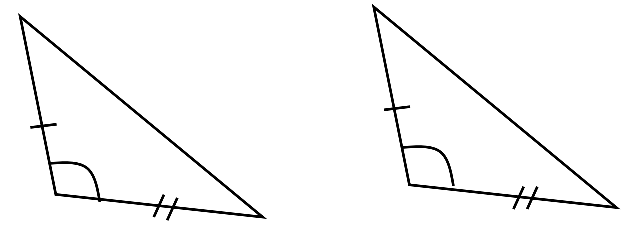
Condition 1

Two triangles are congruent if three sides of one triangle are equal in length to the three sides of the other triangle. (SSS)



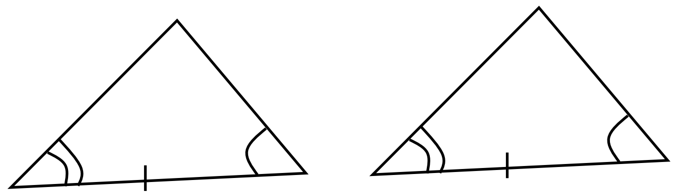
Condition 2

Two triangles are congruent if two sides and the included angle are equal to two sides and the included angle of the other triangle. (SAS)



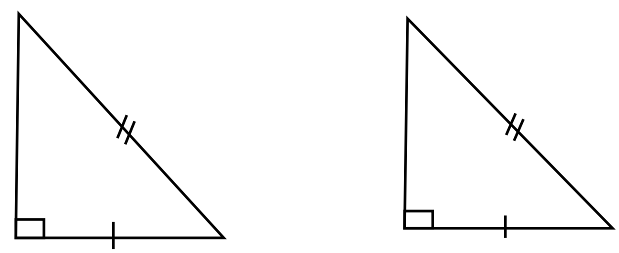
Condition 3

Two triangles are congruent if two angles and one side of a triangle are equal to two angles and a corresponding side of the other triangle. (AAS)

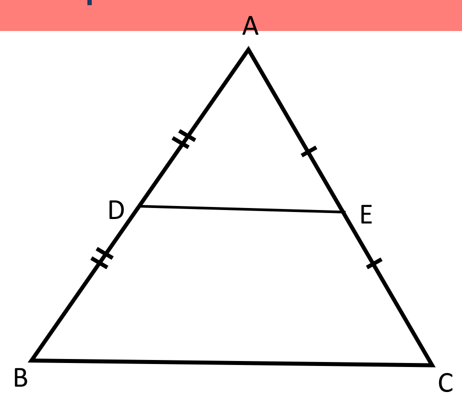


Condition 4

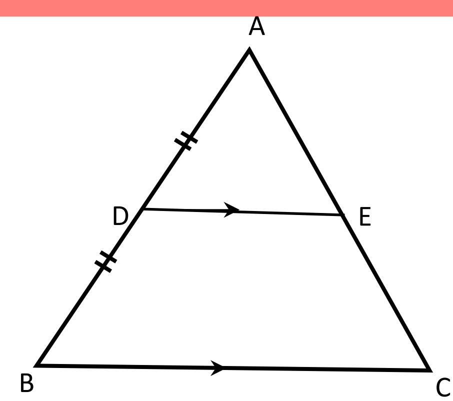
Two right-angled triangles are congruent if the hypotenuse and a side of the one triangle is equal to the hypotenuse and a side of the other triangle. (RHS)



The Midpoint Theorem



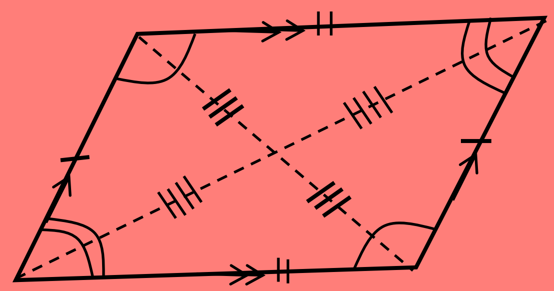
If $AD = DB$ and $AE = EC$, then $DE // BC$ and $DE = \frac{1}{2}BC$



If $AD = DB$ and $DE // BC$, then $AE = EC$ and $DE = \frac{1}{2}BC$

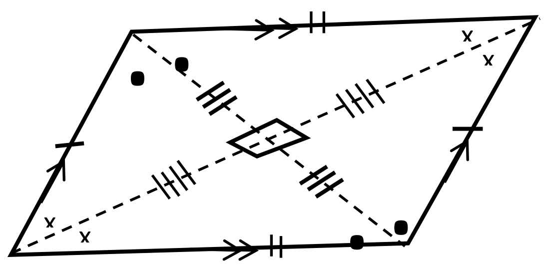
Properties of Quadrilaterals: (Properties of quadrilaterals and their application are important in solving Euclidean Geometry problems).

Parallelogram (Parm)



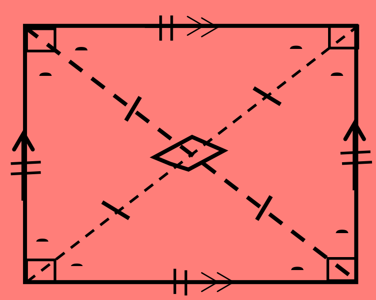
- Opposite sides are parallel and equal in length
- Opposite angles are equal
- Diagonals bisect each other
- Area = base x perpendicular height

Rhombus



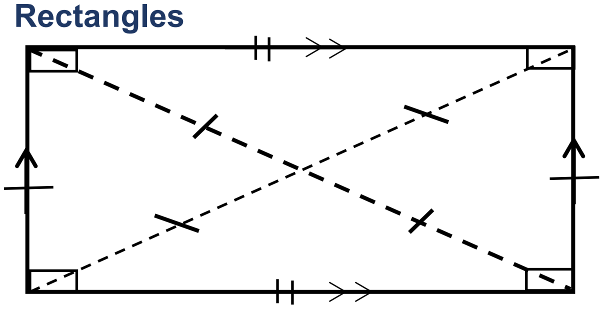
- All sides are equal in length
- Opposite sides are parallel
- Opposite angles are equal
- Diagonals bisect each other at $90°$
- Diagonals bisect the corner angles
- Area = $\frac{1}{2}$ (diagonal 1 x diagonal 2)

Square

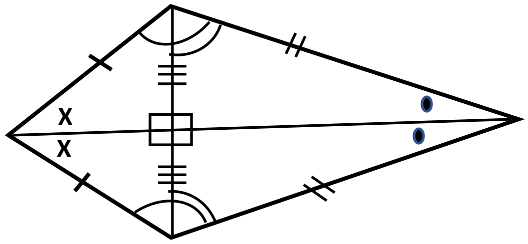
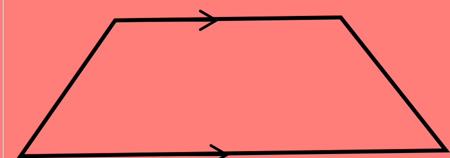


- All sides are equal in length
- Opposite sides are parallel
- Corner angles equal $90°$
- Diagonals are equal and bisect each other at $90°$
- Diagonals bisect the corner angles
- Area = side x side

Rectangles

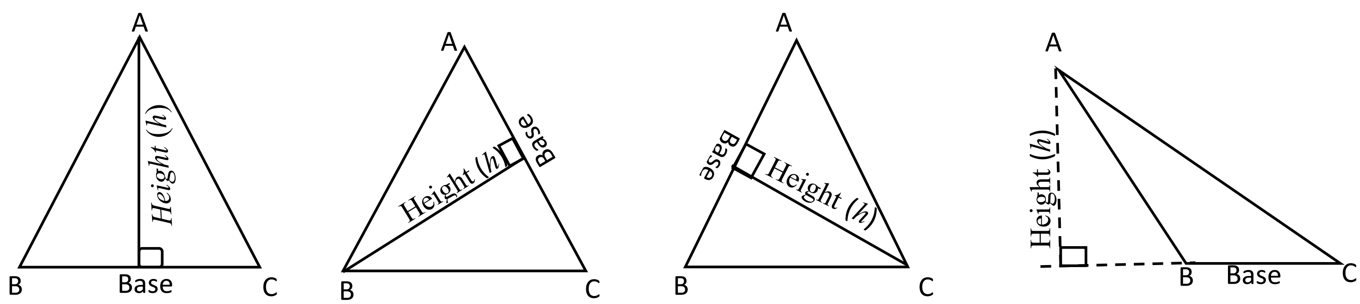


- Opposite sides are parallel and equal in length
- Corner angles equal $90°$
- Diagonals are equal and bisect each other
- Area = base x height

Kite 	• Two pairs of adjacent sides are equal • A single pair of opposite angles are equal • Diagonals intersect each other at $90°$ • One diagonal bisects the corner angle • Shorter diagonal is bisected by the longer diagonal • Area = $\frac{1}{2}$ (diagonal1 x diagonal2)
Trapezium 	• At least one pair of opposite sides are parallel • Area = $\frac{1}{2}$ (sum of 2 //sides) x height

Area of Triangle

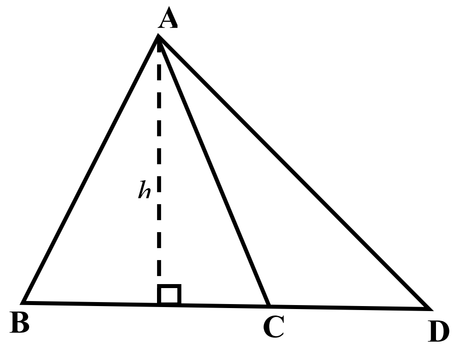
(a) The height or altitude of a triangle is always relative to the chosen base.



In all cases, the area of the triangles can be calculated by using the formula

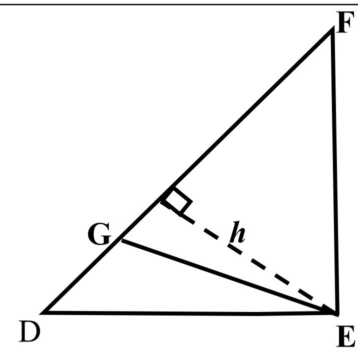
Area of $\Delta ABC = \frac{1}{2}(base) \times (height)$

(b) Two triangles which share a common vertex have a common height.



$$Area\ \Delta ABC = \frac{1}{2}.BC.h$$

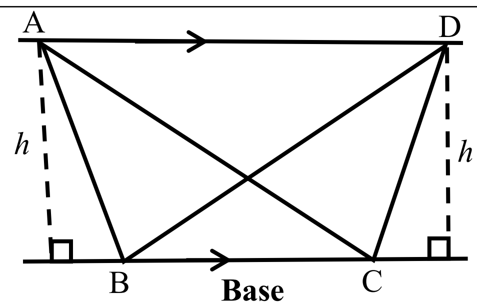
$$Area\ \Delta ACD = \frac{1}{2}.CD.h$$

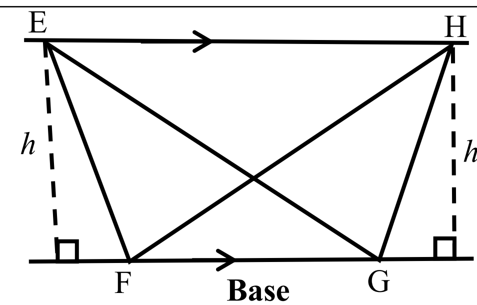


$$Area\ \Delta DEG = \frac{1}{2}.\ DG.h$$

$$Area\ \Delta FGE = \frac{1}{2}.\ GF.h$$

(c) Triangles with equal or common bases lying between parallel lines have the same area

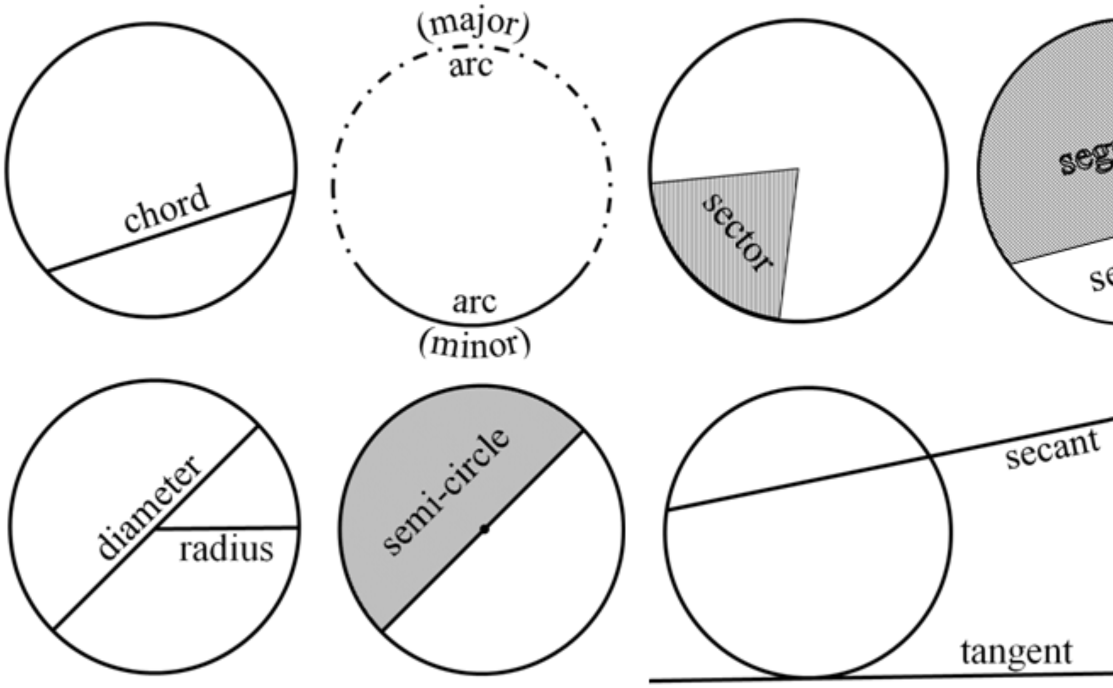




GRADE 11-12 EUCLIDEAN GEOMETRY

NOTE: Grade 11 Geometry is very important as it is examinable in full with the Grade 12 Geometry. The **nine circle** geometry theorems must be understood and mastered in order to achieve success in solving riders.

KEY CONCEPTS



Proofs of the following theorems are examinable:

- The line drawn from the centre of a circle perpendicular to a chord bisects the chord;
- The line drawn from the centre of a circle to the meet point of the chord is perpendicular to the chord;
- The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre);
- The opposite angles of a cyclic quadrilateral are supplementary;
- The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment;
- A line drawn parallel to one side of a triangle divides the other two sides proportionally;
- Equiangular triangles are similar

The line drawn from the centre of a circle perpendicular to a chord bisects the chord

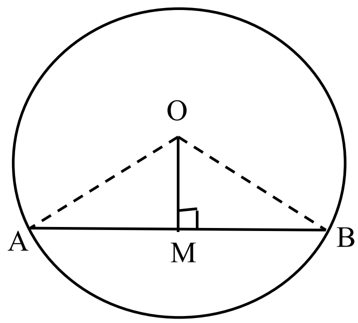
GIVEN: The line drawn from the centre of a circle perpendicular to a chord

R.T.P. bisects the chord

GIVEN: Circle O and $OM \perp AB$

R.T.P. : AM = MB

Construction: Draw radii OA and OB



Proof:

In ΔOAM and ΔOBM,

OA = OB . . . (Radii)

OM = OM . . . (Common)

$O\hat{M}A = O\hat{M}B$ . . . (Each = $90°$)

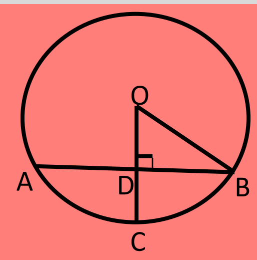
$\therefore \Delta OAM \equiv \Delta OBM$. . . (RHS)

$\therefore$ AM = MB . . . (From congruency)

NOTE: Conversely, a line segment drawn from the centre of a circle to the midpoint of a chord, is perpendicular to the chord

Example:

Given: Circle with centre O and chord AB. OC $\perp$ AB, cutting AB at D, with C on the circumference. OB = 13 units and AB = 24 units. Calculate the length of CD.



AD = DB . . . (Line from centre $\perp$ chord)

But AB = 24 units . . . . (Given)

$\therefore$ DB = 12 units

In ΔODB,

$OB^2 = OD^2 + DB^2$ . . .(Pythagoras)

$13^2 = OD^2 + 12^2$

$OD^2 = 13^2 - 12^2$

OD = $\sqrt{169 - 144}$

= 5 units

But OB = OC = 13 units . . . . (Radii)

And $CD = OC - OD =$ 13 – 5

= 8 units

The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre)

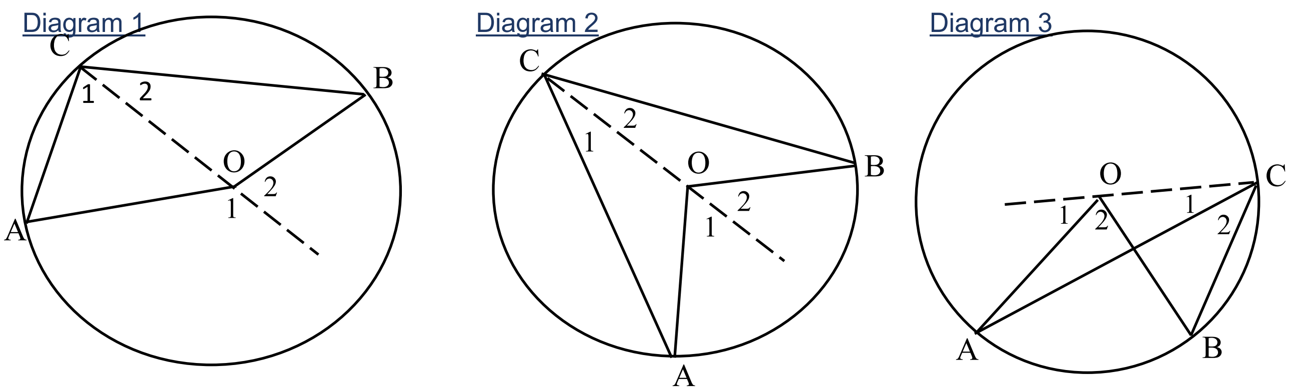
GIVEN: The angle subtended by an arc at the centre of a circle
R.T.P. is double the size of the angle subtended by the same arc at the circle

FORMAL PROOF:

Given : Circle with centre O and arc AB subtending $A\hat{O}B$ at the centre and $A\hat{C}B$ at the circle

R.T.P : $A\hat{O}B = 2 \times A\hat{C}B$

Construction : Draw CO and produce



Proof:

$\hat{O}_1 = \hat{C}_1 + \hat{A}$ . . . . (Ext $\angle^s$ of Δ = sum of int. opp $\angle s$)

But $\hat{C}_1 = \hat{A}$ . . . . ($\angle s$ opp = sides OA and OC radii)

$\therefore\ \hat{O}_1 = 2\hat{C}_1$

similarly $\hat{O}_2 = 2\hat{C}_2$

In Diagram 1 & 2: $\hat{O}_1 + \hat{O}_2 = 2(\hat{C}_1 + \hat{C}_2)$ $\therefore\ A\hat{O}B = 2 \times A\hat{C}B$

In Diagram 3: $\therefore\ \hat{O}_2 - \hat{O}_1 = 2(\hat{C}_2 - \hat{C}_1)$ $\therefore\ A\hat{O}B = 2 \times A\hat{C}B$

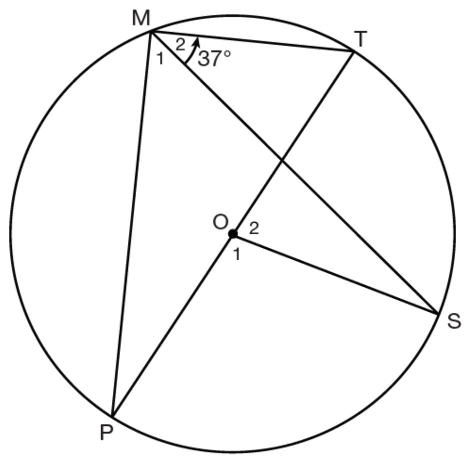
The inscribed angle subtended by the diameter of a circle at the circumference is a right angle. **(∠ in a semi-circle).**

In the diagram alongside, PT is a diameter of the circle with centre O. M and S are points on the circle on either side of PT. MP, MT, MS and OS are drawn. $\hat{M} = 37°$.

Calculate, with reasons, the size of:

a) $\hat{M}_1$

b) $\hat{O}_1$



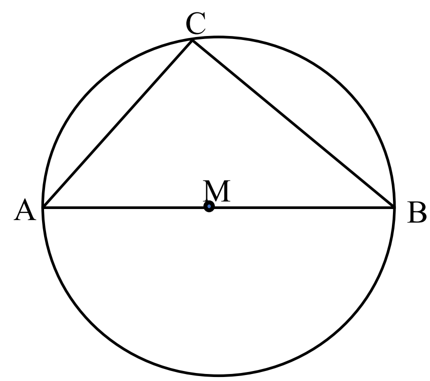
SOLUTION:

a) $P\hat{M}T = 90°$ **. . . . (** $\angle s$ in a semi - circle **)**

$\hat{M}_1 = 90° - 37°$

$\hat{M}_1 = 53°$

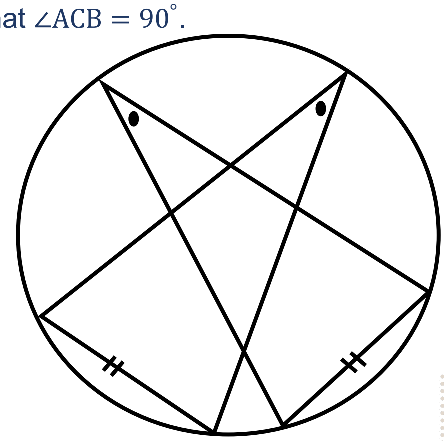
b) $\hat{O}_1 = 2(53°) = 106°$ **. . . . (** $\angle$ at centre $= 2 \times \angle$ at circumference)



NOTES:

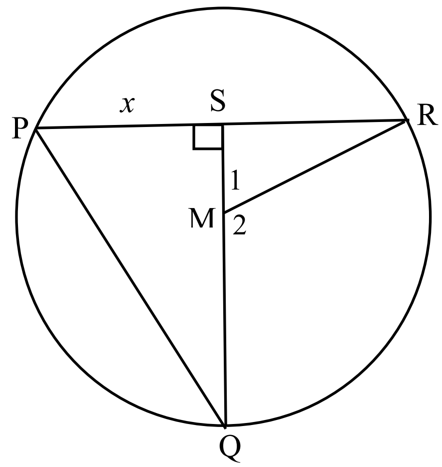
a). If, for any circle with centre M, point B moves in an anticlockwise direction, it reaches a point where arc AB becomes a diameter of the circle. In that case, arc AB subtends ∠AMB at the centre and ∠ACB at the circumference. Using the above theorem and the fact that ∠AMB is a straight angle, it can be deduced that $\angle ACB = 90°$.

b). Equal chords subtend equal angles at the centre and at the circumference.



Example:

In the accompanying diagram, PR and PQ are equal chords of the circle with centre M. QS $\perp$ PR at S. PS = x units and MR is drawn.



a). Express, with reasons, QS in terms of x. (5)

b). If $x = \sqrt{12}$ units and MS = 1 unit, calculate the length of the radius of the circle. (2)

c). Calculate, giving reasons, the size of $\angle P$. (5)

SOLUTION:

a). PS = SR = x .... (Line from centre $\perp$ chord)

$\therefore$ PR = PQ = 2x .... (Equal chords, given)

In ΔPQS,

$PQ^2 = QS^2 + PS^2$ .... (Pythagoras)

$QS^2 = (2x)^2 - x^2$

$= 3x^2$

QS = $\sqrt{3}\,x$ units

b). Radius = QS – SM

$= \sqrt{3}\,x - 1$

$= \sqrt{3}\sqrt{12} - 1$

$= 6 - 1$

$= 5$ units

c). $\tan P = \frac{QS}{PS} = \frac{6}{2\sqrt{3}}$

$= \frac{3}{\sqrt{3}}$

$= \sqrt{3}$

$\therefore \hat{P} = 60°$

The opposite angles of a cyclic quadrilateral are supplementary

Note that all 4 vertices of a quadrilateral must lie on the same circle for the quadrilateral to be cyclic.

GIVEN: The opposite angles of a cyclic quadrilateral

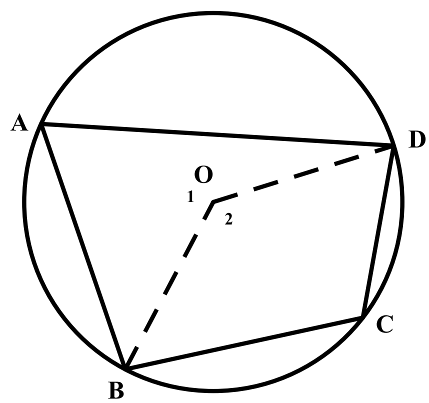
R.T.P. are supplementary

Given any circle with centre O, passing through the vertices of cyclic quadrilateral ABCD

R.T.P.: $\hat{A}+\hat{C}=180°$ and $\hat{B}+\hat{D}=180°$

Construction : Draw BO and OD

Proof:



$\hat{O}_2 = 2\hat{A}$ ($\angle$ at the centre $= 2\times\angle$ at circle)

$\hat{O}_1 = 2\,\hat{C}$ ($\angle$ at the centre $= 2\times\angle$ at circle)

$\hat{O}_1+\hat{O}_2 = 2\,(\hat{A}+\hat{C})$

but $\hat{O}_1+\hat{O}_2 = 360°$ ($\angle$'s around a point)

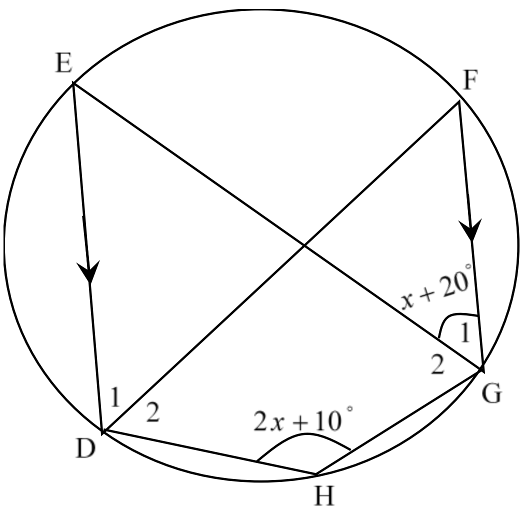
hence $\hat{A}+\hat{C} = 180°$

also $\hat{B}+\hat{D}=180°$ (sum of int $\angle$'s of quad)

Example

D, E, F, G and H are points on the circumference of a circle.

$\hat{G}_1 = x + 20^\circ$ and $\hat{H} = 2x + 10^\circ$ DE || FG.



a) Determine the size of $D\hat{E}G$ in terms of x

b) Calculate the size of $D\hat{H}G$

SOLUTION

a) $D\hat{E}G = 180 - (2x + 10)$ (opposite angles of a cyclic quadrilateral)

$180 - 2x - 10$

$= 170 - 2x$

b) $\hat{G}_1 = \hat{E}$ (alt <s $DE // FG$)

$x + 20 = 170 - 2x$

$3x = 150$

$x = 50^0$

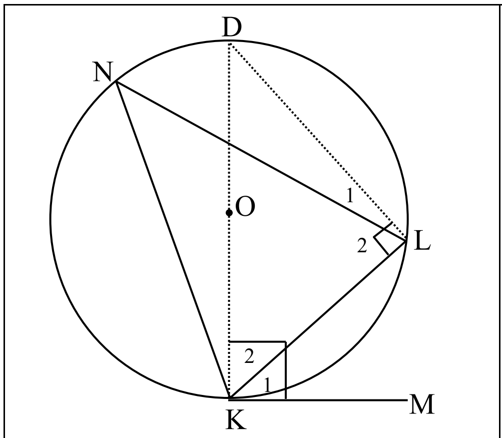
$\therefore D\hat{H}G = 70^0$

Angle between a tangent and a chord is equal to the angle in the alternate segment.

GIVEN: Angle between a tangent and the chord
R.T.P. is equal to the angle in the the alternate segment

GIVEN: $M\hat{K}L$ OR $\hat{K}_1$ with chord KL and tangent MK

RTP: $M\hat{K}L = \hat{N}$



Construction: Draw diameter KOD and join DL

Proof:

$\hat{K}_1 + \hat{K}_2 = 90°$ …..tan ⊥ rad

$\hat{K}_1 = 90° - \hat{K}_2$

$D\hat{L}K = 90°$ …. $\angle s$ at semi-circle

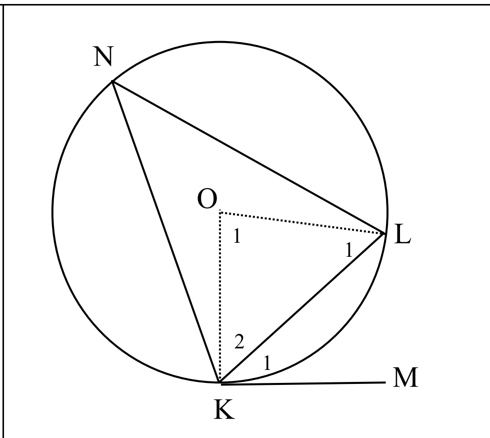
$\hat{K}_2 + \hat{D} = 90°$ ….sum of $\angle s$ in a Δ

$\hat{D} = 90° - \hat{K}_2$

$\hat{D} = \hat{K}_1$ ……. Both $= 90° - \hat{K}_2$

But $\hat{D} = \hat{N}$. . . $\angle s$ in same segment

$\therefore \hat{K}_1 = \hat{N}$ …..both $= \hat{D}$



Construction: Join OL and OK

Proof:

$\hat{K}_1 + \hat{K}_2 = 90°$ . . . tan ⊥ *rad*

$\hat{K}_1 = 90° - \hat{K}_2$

$\hat{O}_1 + \hat{K}_2 + \hat{L}_1 = 180°$ . . . sum of $\angle s$ in a Δ

$\hat{K}_2 + \hat{L}_1 = 180° - \hat{O}_1$

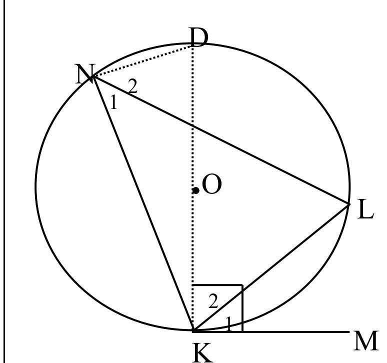
$\hat{O}_1 = 2\hat{N}$ . . . $\angle$ at centre = 2 $\angle$ at circumf.

$\hat{K}_2 + \hat{L}_1 = 180° - 2\hat{N}$

But $\hat{K}_2 = \hat{L}_1$ . . . $\angle s$ opp = sides

$\hat{K}_2 = 90° - \hat{N}$

$\hat{N} = 90° - \hat{K}_2$; $\therefore \hat{K}_1 = \hat{N}$



Construction: Join OK and extend to D on the circumference.
Join ND.

$\hat{K}_1 + \hat{K}_2 = 90°$ . . . tan ⊥ *rad*

$\hat{N}_1 + \hat{N}_2 = 90°$ . . . $\angle s$ $\angle$ in the semi-circle

$\hat{K}_1 + \hat{K}_2 = \hat{N}_1 + \hat{N}_2$ . . both $= 90°$

But $\hat{N}_2 = \hat{K}_2$. . . $\angle s$ in same segment

$\therefore \hat{K}_1 = \hat{N}_1$

PROPORTIONALITY

- It is important to stress to learners that proportion gives no indication of actual length. It only indicates the ratio between lengths.
- Make sure that you know the meaning of ratios. For example, the ratio $\frac{AB}{BC} = \frac{2}{3}$ does not necessarily mean that the length of AB is 2 and the length of BC is 3.

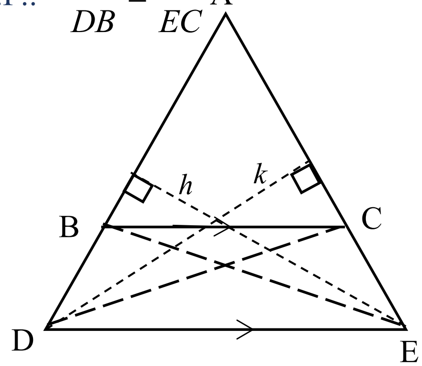
The line drawn parallel to one side of a triangle divides the other two sides proportionally

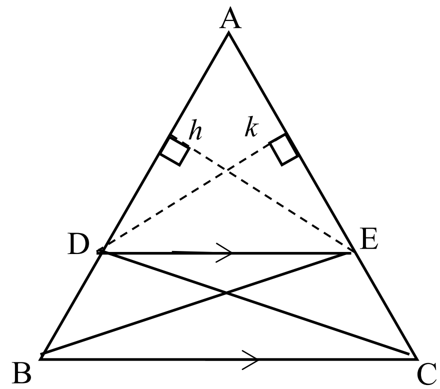
GIVEN: The line drawn parallel to one side of a triangle

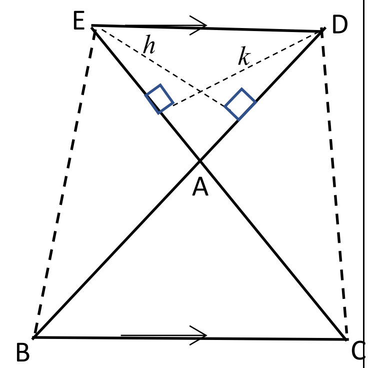
R.T.P. divides the other two sides proportionally

Given : Δ ABC, D lies on AB and E lies on AC. And DE // BC.

R.T.P.: $\dfrac{AD}{DB}=\dfrac{AE}{EC}$







$$\frac{Area\ \Delta ADE}{Area\ \Delta BDE}=\frac{\frac{1}{2}AD\times h}{\frac{1}{2}DB\times h}=\frac{AD}{DB}\ (same\ height)$$

$$\frac{Area\ \Delta ADE}{Area\ \Delta CED}=\frac{\frac{1}{2}AE\times k}{\frac{1}{2}EC\times k}=\frac{AE}{EC}\ (same\ height)$$

Make sure the height used corresponds with the correct base as indicated in the construction

But $Area\ \Delta BDE = Area\ \Delta CED$ (same base and between // lines)

$$\therefore \frac{Area\ \Delta\ ADE}{Area\ \Delta\ BDE}=\frac{Area\ \Delta\ ADE}{Area\ \Delta CED}$$

$$\therefore \frac{AD}{DB}=\frac{AE}{EC}$$

SIMILARITY THEOREM:

- Know the conditions under which two triangles are similar
- Note that to prove that sides are in proportion, similarity of triangles is proved and **not congruency**
- When proving that the two triangles are similar, make sure that the equal angles correspond: i.e. if given that $\Delta ABC /// \Delta BDA$ then you cannot say that $\Delta ABC /// \Delta ABD$
- To prove triangles are similar, we need to show that two angles (AAA) are equal OR three
  sides in proportion (SSS).
- The examples on similar triangles illustrate a highly systematic and effective strategy which has been used in the teaching of triangle geometry.

Equiangular triangles are similar

GIVEN: Equiangular triangles

R.T.P. are similar

In ΔAXY and ΔDEF

$AX = DE \quad (construction)$

$AY = DF \quad (construction)$

$\hat{A} = \hat{D} \quad (given)$

$\therefore \Delta AXY \equiv \Delta DEF \qquad (SAS)$

$A\hat{X}Y = \hat{E}$ **but** $\hat{E} = \hat{B} \quad$ (given)

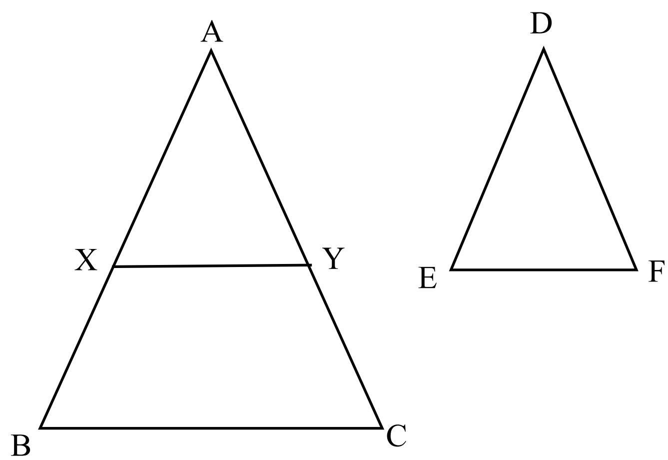
$\therefore A\hat{X}Y = \hat{B}$

$\Rightarrow XY // BC \ (corresponding \angle s =)$

$\frac{AB}{AX} = \frac{AC}{AY} \quad$ (line // one side of Δ)

but $AX = DE$ and $AY = DF \quad (construction)$

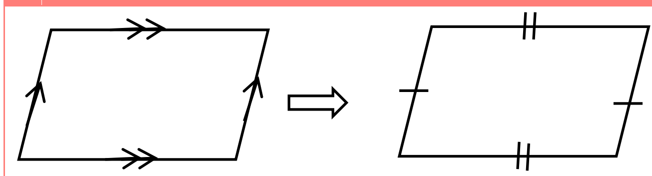
$\therefore \frac{AB}{DE} = \frac{AC}{DF}$



Similarly by marking off equal lengths on BA and BC, it can be shown that: $\frac{AB}{DE} = \frac{BC}{EF}$

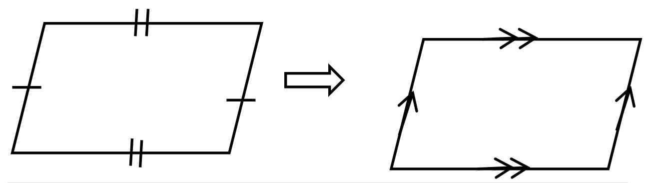
$\therefore \frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$

THEOREMS AND THEIR CONVERSES



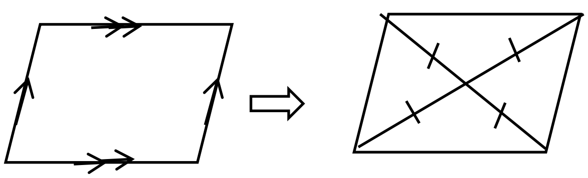
(Opp. Sides parm =)

If given a parallelogram, then the opposite sides of the parm are equal



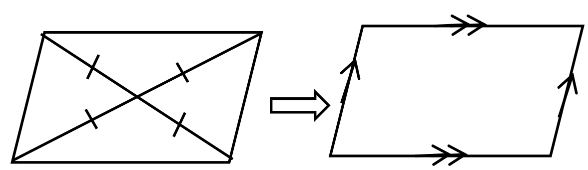
(Conv. Opp. Sides parm)

If opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram



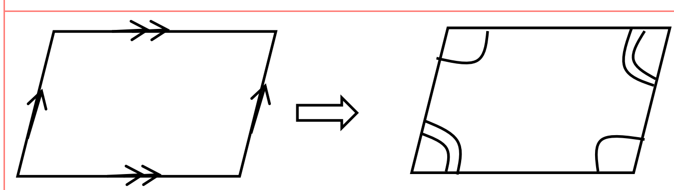
(diags. parm)

If given a parallelogram, then the diagonals bisect each other

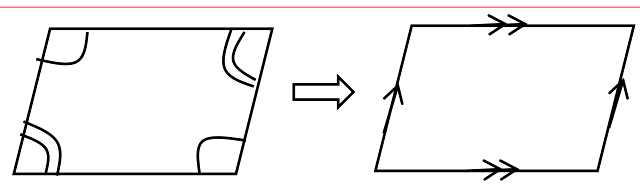


(Conv. diags. parm)

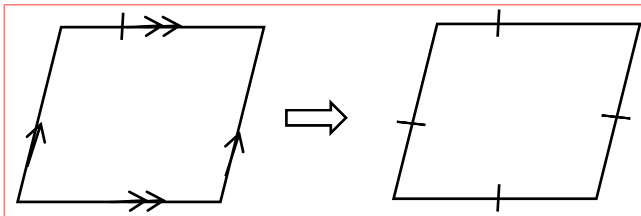
If diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram



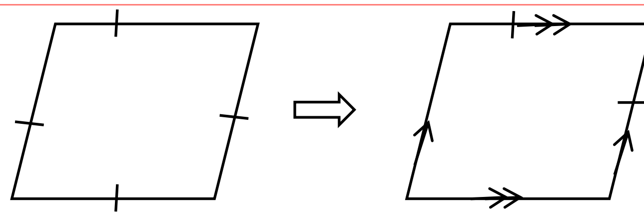
(opp. $\angle$'s parm)



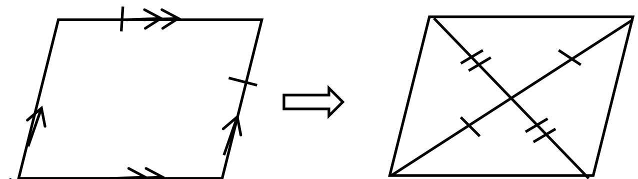
(conv. opp. $\angle$'s parm)



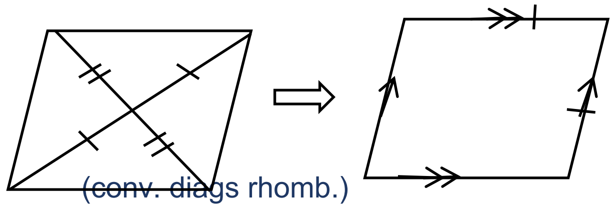
(Sides of rhomb.)



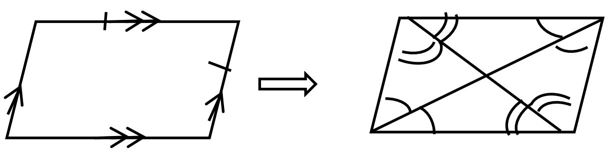
(conv. sides of rhomb.)



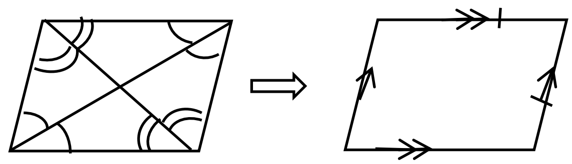
(diags rhomb.)



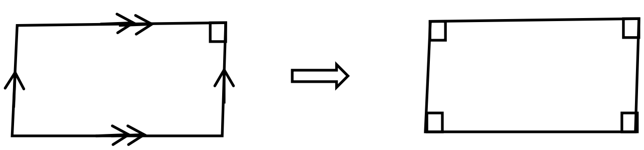
(conv. diags rhomb.)

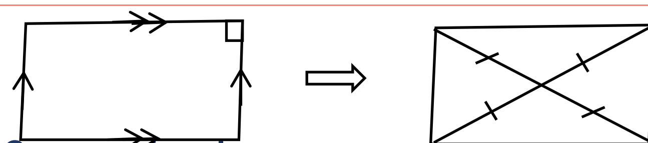


(diags rhomb.)



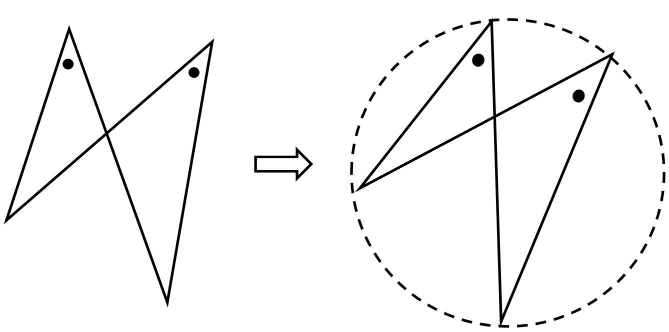
(conv. diags rhomb.)

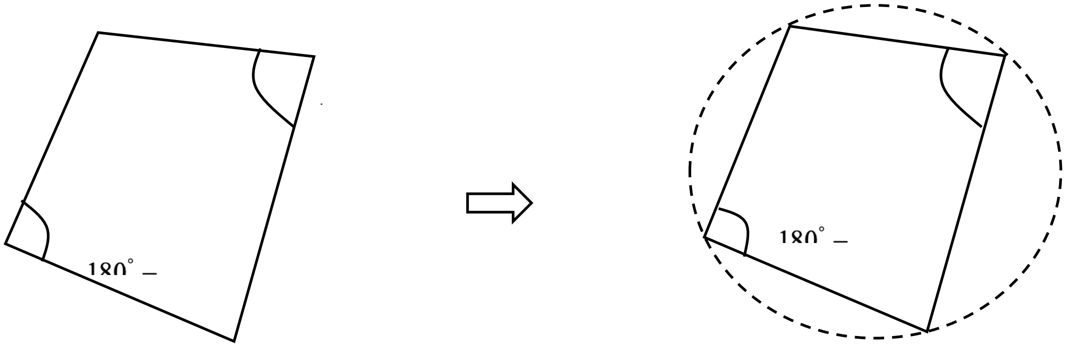




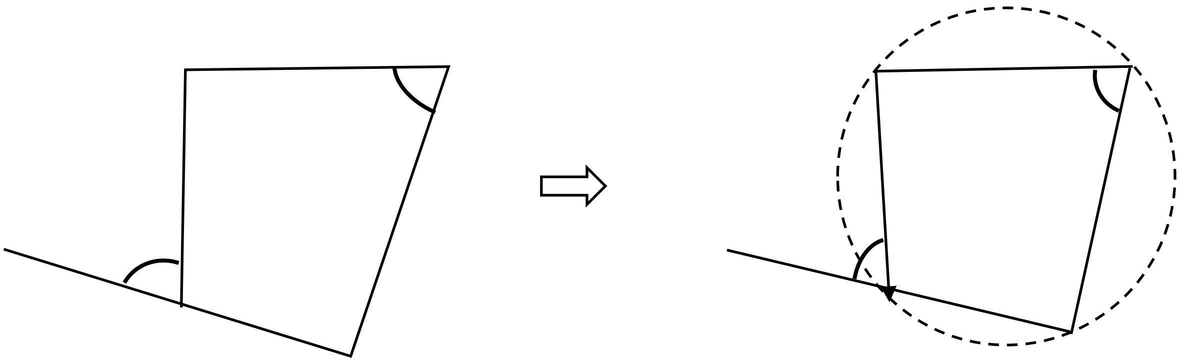
Conv. rectangle

Conv. $\angle$'s in the same segment





Conv. opp. $\angle's$ **cyclic quad**



(conv. ext, $\angle's$ **cyclic quad.)**

NOTE: ***success in answering Euclidean Geometry comes from regular practice, starting off with the easy and progressing to the difficult***.

Important points about solving riders in Geometry

1 Read the problem carefully for understanding. You may need to underline important points and make sure you understand each term in the given and conclusion. Highlight key word like centre, diameter, tangent, because they are linked to theorems you would need to solve riders.

2 Draw the sketch if it is not already drawn. The sketch need not be accurately drawn but must as close as possible to what is given i.e. lines and angles which are equal must look equal or must appear parallel etc. Also indicate further observations based on previous theorems.

3 Indicate on the figure drawn or **given** all the equal lines and angles, lines which are parallel, drawing in circles, measures of angles given if not already indicated in the question. Put in the diagram answers that you get as you work along the question; you may need to use them as you work along the question. It might be more helpful to have a variety of colour pens or highlighters for this purpose.

4 Usually you can see the conclusion before you actually start your **formal proof** of a rider. Always write the reason for each important statement you make, quoting in brief the theorem or another result as you proceed.

5 When proving similar triangles, the triangles are already similar, you just need to provide reasons for similarity. It helps to highlight the two triangles so that it will be easy to see why corresponding angles are equal. Do not forget to indicate the reason for similarity that is AAA or $\angle\angle\angle$.

6 Answers must be worked out sequentially, there's always a way out.

7 Sometimes you may need to work backwards, asking yourself what I need to show to prove this **conclusion** (required to be proved) and then see if you can prove that as you reverse. **NB, do not use answer/ what is supposed to prove in the proof.**

9 **WRITE GEOMETRY REASONS CORRECTLY. Refer to acceptable reasons as reflected in the Examination Guidelines.**

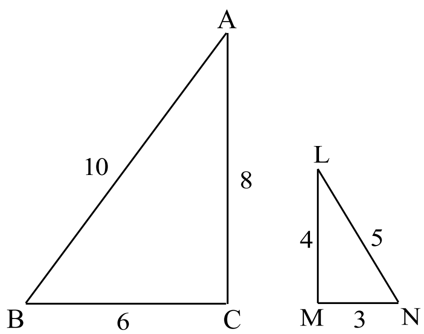
2.1 PRACTICE EXERCISES

- Diagrams are not drawn to scale
- Refrain from making assumptions. (For example, if a line looks like a tangent, but no tangent is mentioned in the description statement, the three theorems associated with a tangent cannot be applied. Sometimes the examiner may want you to prove that it is a tangent)

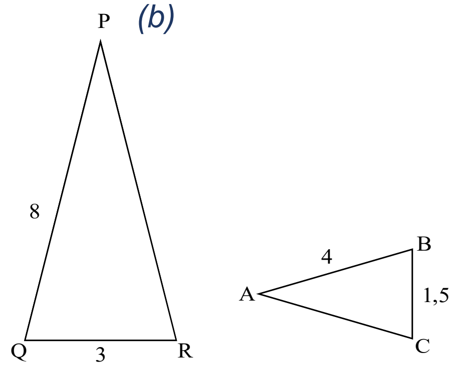
QUESTION 1

Are the following pairs of triangles similar? Give a reason for your answer.

(a)



(b)



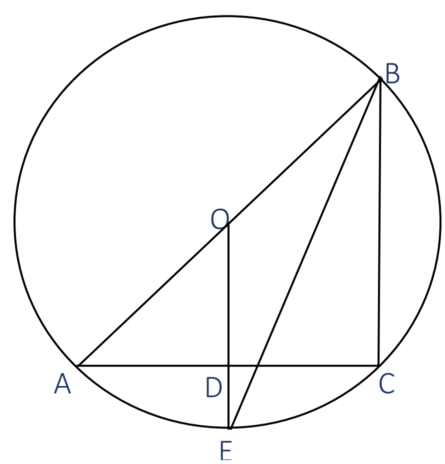
<u>QUESTION 2</u>

In the accompanying figure, AOB is a diameter

of the circle AECB with centre O.

OE // BC and OE meets AC at D.

B and E are joined.



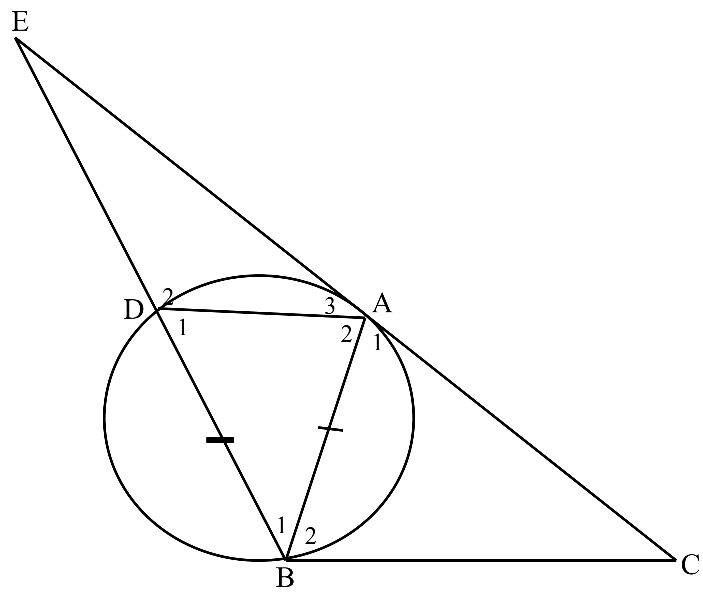
2.1 Prove that AD = DC

2.2 Prove that EB bisects $A\hat{B}C$

2.3. If $E\hat{B}C = x$, express $B\hat{A}C$ in terms of *x*.

QUESTION 3

In the diagram alongside, BC and CAE are tangents to circle DAB and BD = BA.



3.1 Prove that

3.1.1 $\hat{D}_2 = \hat{A}_2 + \hat{A}_3$

3.1.2 DA // BC

3.2 Hence, deduce that $\frac{ED}{AB} = \frac{EA}{AC}$

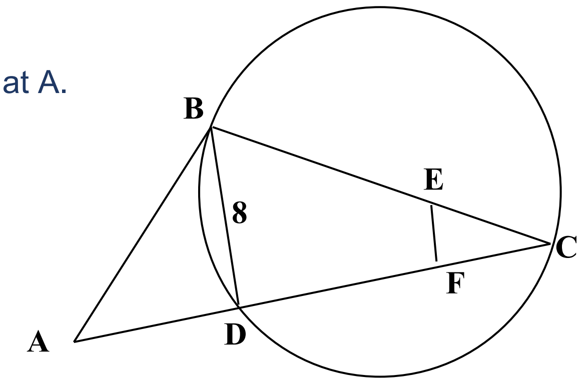
3.3 Calculate the length of AB, if it is further given that EC : EA = 5 : 2 and ED = 18 units.

3.4 Prove that Δ EDA ||| Δ EAB.

QUESTION 4

In the diagram, BC = 17 units, where BC is a diameter of the circle. The length of the chord BD is 8 units.

The tangent at B meets CD produced at A.



4.1 Calculate, with reasons, the length of DC

4.2 E is a point on BC such that BE : EC = 3 : 1. EF is parallel to BD with F on DC.

4.2.1 Calculate, with reasons, the length of CF

4.2.2 Prove that ΔBAC /// ΔFEC

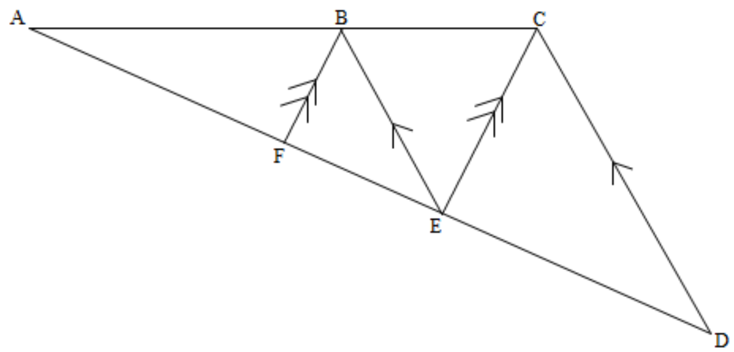
4.2.3 Calculate the length of AC

QUESTION 5

In ΔADC, E is a point on AD and B is a point on AC such that EB//DC.

F is a point on AD such that FB//EC.

It is also given that AB = 2BC



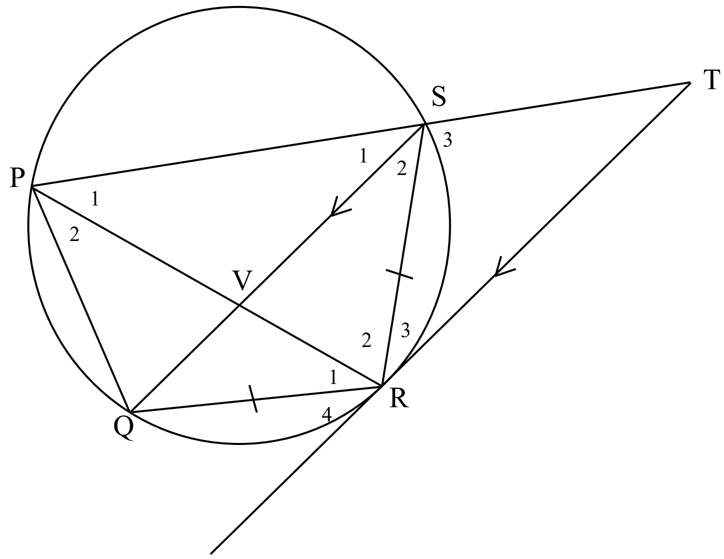
5.1 Determine the value of AI

5.2 Calculate the length of ED if AF = 8cm

QUESTION 6

In the accompanying figure, PQRS is a cyclic quadrilateral with RS = QR. A straight line (not given as a tangent) through R, parallel to QS, meets PS produced at T. P and R are joined.

If $\hat{R}_3 = x$



6.1 Prove giving reasons that RT is a tangent to the circle at R

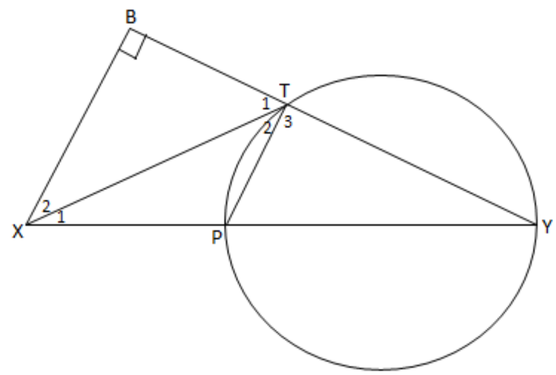
6.2 Prove that $\hat{R}_1 = \hat{T}$

6.3 Prove that ΔRST /// ΔPQR

6.4 If PQ = 4cm and ST = 9cm, Calculate the length of QR

QUESTION 7

In the figure PY is a diameter of the circle and X is on YP produced. XT is a tangent the circle at T and XB is perpendicular to YT produced.

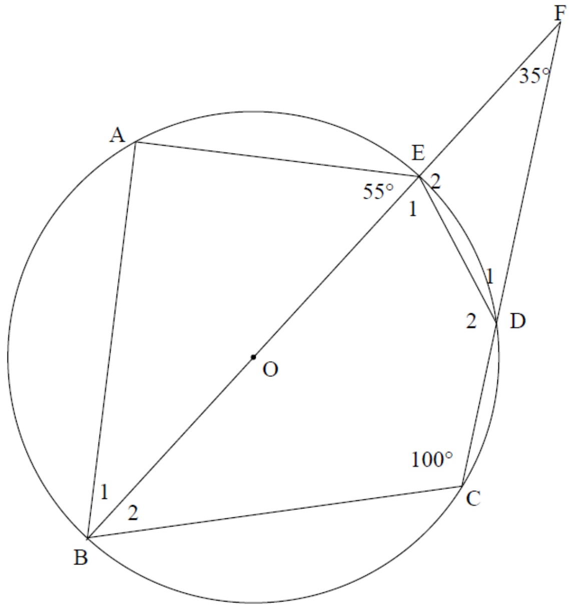


7.1 Prove that BX // TP

7.2 Prove that $\frac{XB}{YB} = \frac{XT}{YX}$

QUESTION 8 (WC 2016 Trial)

In the diagram, O is the centre of the circle. A, B, C, D and E are points on the circumference of the circle. Chords BE and CD produced meet at F. $\hat{C} =$ 100°, $\hat{F} = 35°$ and $A\hat{E}B = 55°$.



8.1 Calculate, giving reasons, each of the following angles:

8.1.1 $\hat{A}$ (2)

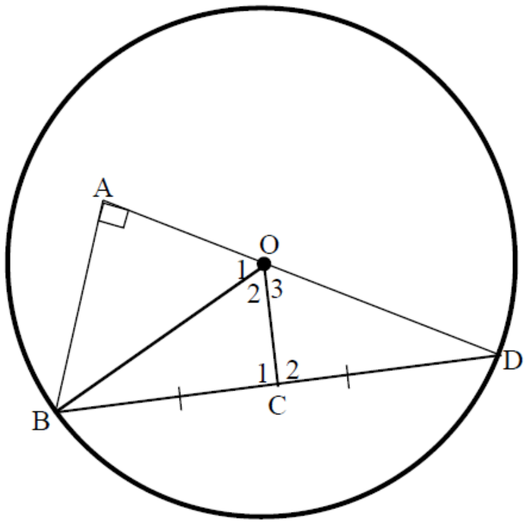
8.1.2 $\hat{E}_1$ (2)

8.1.3 $\hat{D}_1$ (2)

8.2 Prove, giving reasons, that $AB \parallel CF$. (4)

QUESTION 9 (GDE, 2017 Trial)

In the diagram below, O is the centre of the circle. C is the midpoint of chord BD. Point A lies within the circle such that BA $\perp AOD$.



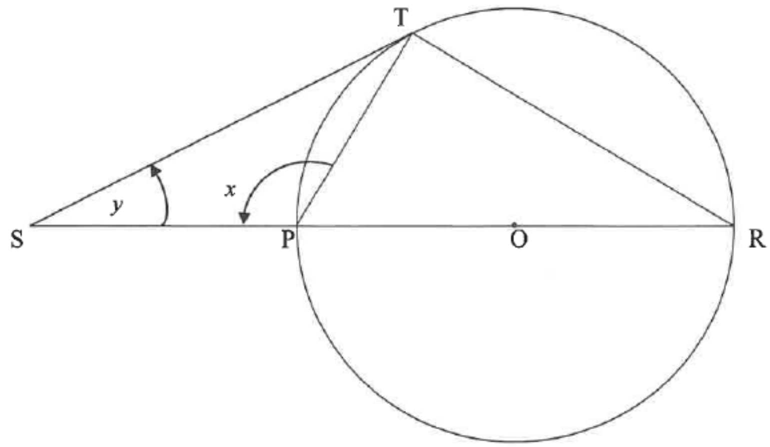
9.1 Show that DA.OD = $OD^2 + OD.OA$. (1)

9.2 Prove that $2DC^2 = OD^2 + OD.OA$ (7)

[8]

QUESTION 10

In the diagram, PR is a diameter of the circle with centre O. ST is a tangent to the circle at T and meets RP produced at S. $S\hat{P}T = x$ and $\hat{S} = y$.

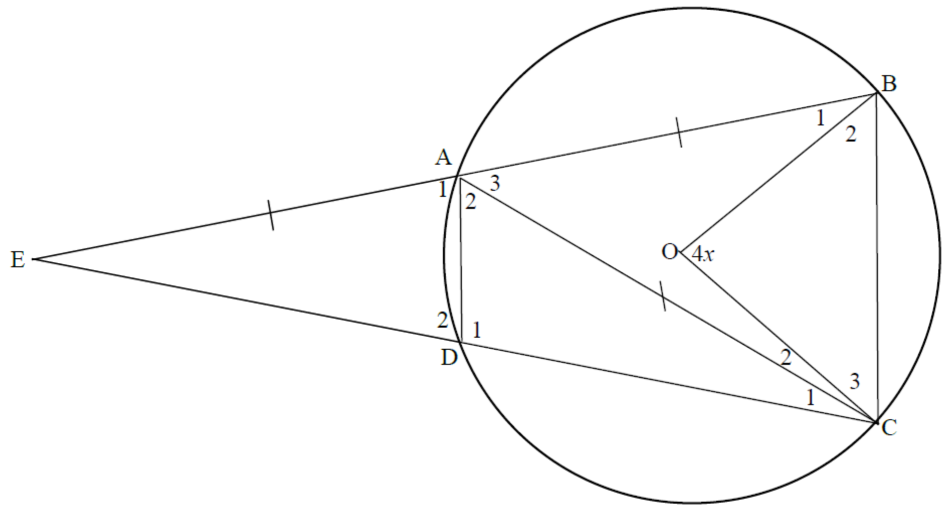


Determine, with reasons, *y* in terms of *x*. **[6]**

QUESTION 11 (GDE, 2018 TRIAL)

In the diagram below, O is the centre of the circle. ABCD is a cyclic quadrilateral. BA and

CD are produced to intersect at E such that AB = AE = AC.



11.1 Determine each of the following angles in terms of *x*:

11.1.1 $\hat{B}_2$ (2)

11.1.2 $\hat{E}$ (5)

11.1.3 C_2 (3)

11.2 If $\hat{E} = \hat{C}_2 = x$, prove that ED is a diameter of circle AED. (4)

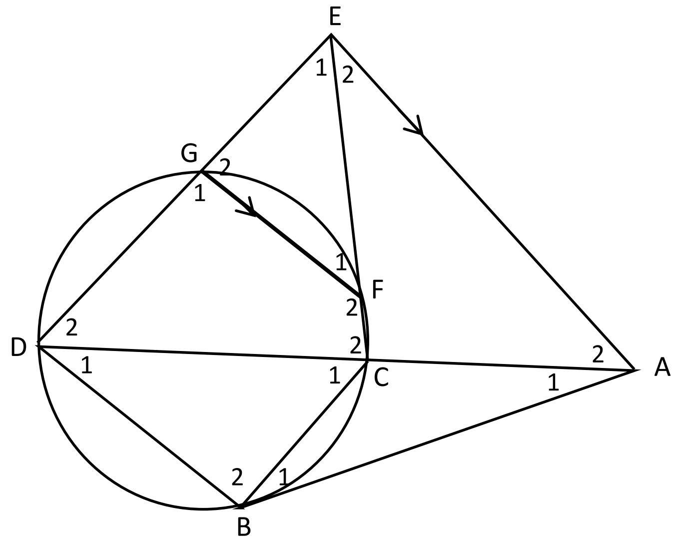
[14]

QUESTION 12 (WC, Sept. 2015)

12.1 Complete the following statement:

If two triangles are equiangular, then the corresponding sides are …

12.2 In the diagram, DGFC is a cyclic quadrilateral and AB is a tangent to the circle at B. Chords DB and BC are drawn. DG and CF produced meet at E and DC is produced to A. EA | | GF.



12.2.1 Give a reason why $\hat{B}_1 = \hat{D}_1$. (1)

12.2.2 Prove that $\Delta ABC\,///\,\Delta ADB$. (3)

12.2.3 Prove that $\hat{E}_2 = \hat{D}_2$. (4)

12.2.4 Prove that $AE^2 = AD \times AC$. (4)

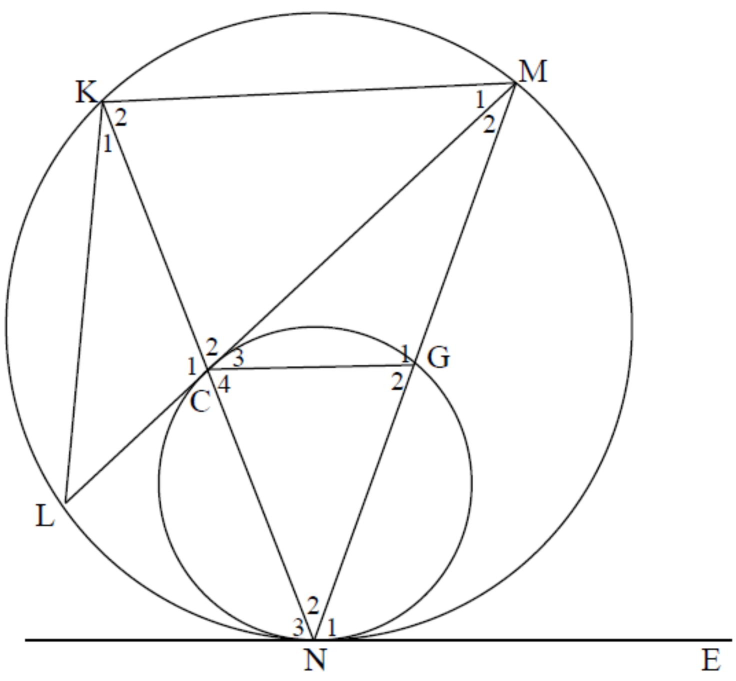
12.2.5 Hence, deduce that AE = AB. (3)

QUESTION 13 (GDE, 2016 Trial)

In the diagram below NE is a common tangent to the two circles. NCK and NGM are double

chords. Chord LM of the larger circle is a tangent to the smaller circle at point C. KL, KM and

CG are drawn.



Prove that:

13.1 $\frac{KC}{KN} = \frac{MG}{MN}$ (4)

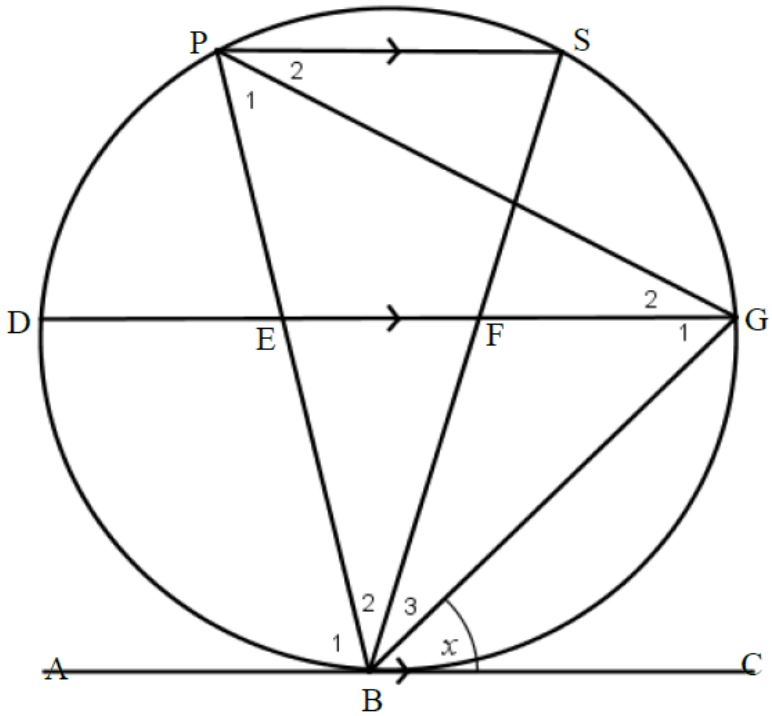
13.2 KMGC s a cyclic quadrilateral if CN = NG. (3)

13.3 $\Delta MCG /// \Delta MNC$. (3)

13.4 $\frac{MC^2}{MN^2} = \frac{KC}{KN}$ (4)

QUESTION 14 (WC, 2016 Trial)

In the diagram, P, S, G, B and D are points on the circumference of the circle such that PS || DG || AC. ABC is a tangent to the circle at B. $G\hat{B}C = x$.



14.1 Give a reason why $\hat{G}_1 = x$. (1)

14.2 Prove that:

14.2.1 $BE = \frac{BP.BF}{BS}$ (2)

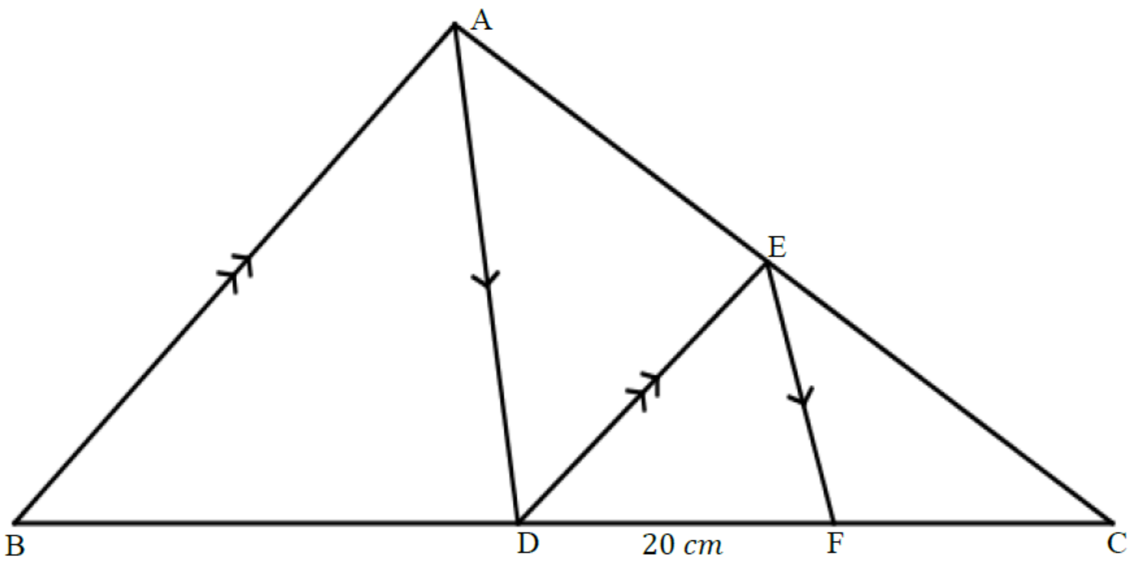
14.2.2 $\Delta BGP /// \Delta BEG$ (4)

14.2.3 $\frac{BG^2}{BP^2} = \frac{BF}{BS}$ (3)

[10]

QUESTION 15 (WC, 2016 Trial)

In the diagram, ΔABC with points D and F on BC and E a point on AC such that EF || AD and DE || BA. Further it is given that $\frac{AE}{EC} = \frac{5}{4}$ and DF = 20 cm.



15.1 Calculate, giving reasons, the length of:

15.1.1 FC (3)

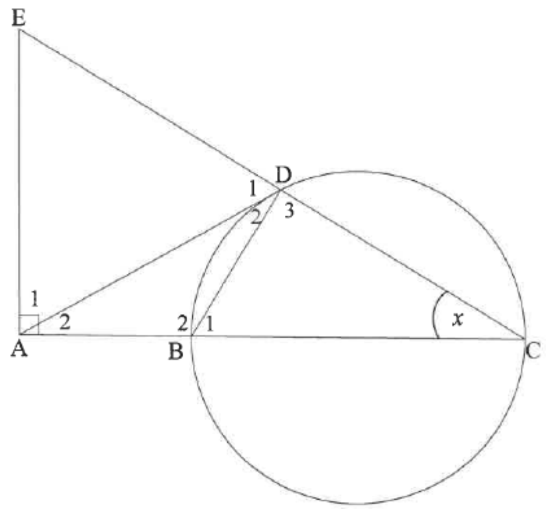
15.1.2 BD (4)

15.2 Determine the following ratio: $\frac{Area\,\Delta ECF}{Area\,\Delta ABC}$ (4)

[11]

QUESTION 16 (DBE, Nov. 2017)

In the diagram, BC is a diameter of the circle. The tangent at point D on the circle meets CB produced at A. CD is produced to E such that EA ⊥ AC. BD is drawn.
Let $\hat{C} = x$.



16.1 Give a reason why:

16.1.1 $\hat{D}_3 = 90°$ (1)

16.1.2 ABDE is a cyclic quadrilateral. (1)

16.1.3 $\hat{D}_2 = x$ (1)

16.2 Prove that:

16.2.1 AD = AE (3)

16.2.2 $\Delta ADB \,|||\, \Delta ACD$ (3)

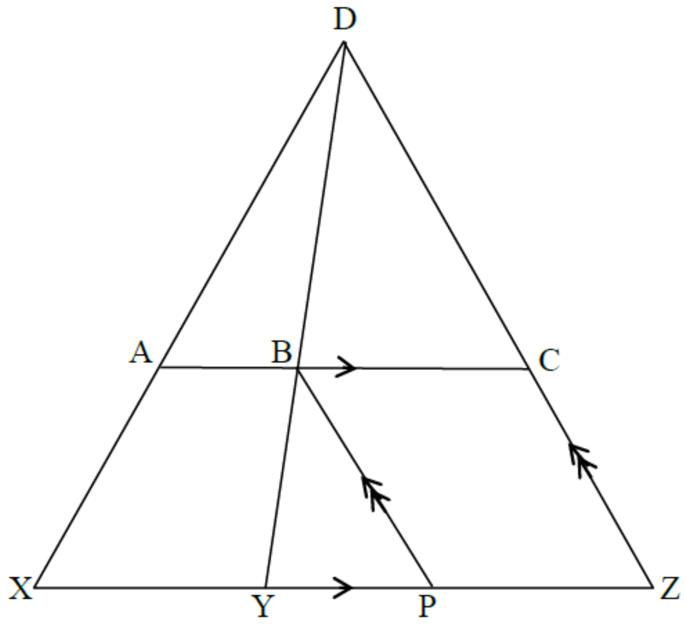
16.3 It is further given that BC = 2AB = 2*r*.

16.3.1 Prove that $AD^2 = 3r^2$. (2)

16.3.2 Hence, prove that ΔADE is equilateral. (4)

QUESTION 17 (GDE, 2018 Trial)

In ΔDXZ below, AC || XZ and BP || DZ. DY is drawn to intersect AC at B.



17.1 Prove that $\frac{BC}{YZ} = \frac{DA}{DX}$ (5)

PRACTICE EXERCISE SOLUTIONS

QUESTION 1

1.1 YES, $\frac{LN}{AB} = \frac{LM}{AC} = \frac{MN}{BC} = \frac{1}{2}$

1.2 NO

QUESTION 2

2.1

$\hat{C} = 90°$ ….. ($\angle$ at semi-circle)

$O\hat{D}A = 90°$…..(corr. $\angle$s = (OE // BC))

AD = DC…..(line segment from centre $\perp$ to chord

bisects the chord)

2.2

$E\hat{B}C = B\hat{E}O$ ……(alt. OE // BC)

but $E\hat{B}O = B\hat{E}O$ ……($\angle$s opp = sides-radii (OE = OB))

$\therefore \quad E\hat{B}C = E\hat{B}O$

$\therefore \quad EB$ bisects $A\hat{B}C$

2.3

$A\hat{O}E = 2A\hat{B}E$ ……. (at centre = 2 × at circumf.)

but $A\hat{B}E = E\hat{B}C = x$...($EB$ bisects $A\hat{B}C$)

$A\hat{O}E = 2x$

$In\ \Delta ABC,\ \ B\hat{A}C + 90° + 2x = 180°$

$B\hat{A}C = 90° - 2x$

QUESTIONS 3

3.1.1

$\hat{A}_3 = \hat{B}_1$ (tan chord)

$\hat{D}_2 = \hat{A}_2 + \hat{B}_1$ (exterior $\angle$ of a Δ)

$\therefore \hat{D}_2 = \hat{A}_2 + \hat{A}_3$

3.1.2

$\hat{B}_2 = \hat{D}_1$ (tan chord)

$\hat{D}_1 = \hat{A}_2$ ($\angle^s$ opposite = sides)

$\therefore \hat{B}_2 = \hat{A}_2$

$\therefore$DA // BC (alternate $\angle^s$=)

3.2 $\frac{ED}{DB}=\frac{EA}{AC}$ (proportionality theorem)

But DB = AB (given)

$\therefore \frac{ED}{AB}=\frac{EA}{AC}$

3.3 EC:EA = 5:2

$\frac{EC}{EA}=\frac{5}{2}$

$\frac{EA+AC}{EA}=\frac{5}{2}$

$1+\frac{AC}{EA}=\frac{5}{2}$

$\frac{AC}{EA}=\frac{3}{2}$

$\frac{EA}{AC}=\frac{2}{3}$

$\therefore \frac{ED}{AB}=\frac{EA}{AC}$

$\frac{18}{AB}=\frac{2}{3}$

AB = 27 *units*

3.4

In Δ EDA and Δ EAB

$\hat{D}_2=\hat{A}_2+\hat{A}_3$ (proved)

$\hat{A}_3=\hat{B}_1$ (tan chord)

$\hat{E}=\hat{E}$ (common)

∴ Δ EDA ||| Δ EAB (∠ ∠ ∠)

QUESTION 4

4.1

$B\hat{D}C=90^{\circ}.....(\angle\, in\, semi-circle)$

$BC^2=DC^2+DB^2\quad (Pythagoras\ theorem)$

$DC^2=17^2-8^2$

$DC=15$

4.2.1

$\frac{CF}{CD}=\frac{CE}{CB}\quad (line\,//\,one\,side\,of\ \Delta)$

$\frac{CF}{15}=\frac{1}{4}$

$4CF=15$

$\therefore CF=3{,}75$

4.2.2

$A\hat{B}C = 90^\circ$.....(tan ⊥ rad)

$B\hat{D}C = 90^\circ$.....(∠ in semi − circle)

$E\hat{F}C = B\hat{D}C = 90^\circ$.....(Corr ∠s, EF // BD)

In ΔBAC and ΔFEC

$\hat{C} = \hat{C}$ ...... (common)

$A\hat{B}C = E\hat{F}C = 90^\circ$ (proven above)

$B\hat{A}C = F\hat{E}C$ (3^{rd} ∠ of Δ)

∴ ΔBAC /// ΔFEC...(AAA)

4.2.3

$$EC = \frac{1}{4} \times 17 = 4{,}25$$

$$\frac{AC}{EC} = \frac{BC}{FC} \quad (\Delta BAC\ ///\ \Delta FEC)$$

$$\frac{AC}{4{,}25} = \frac{17}{3{,}75}$$

$$AC = 19{,}27$$

QUESTION 5

5.1

$$\frac{AF}{FE} = \frac{2}{1}$$

$$FE = \frac{AF}{2} = \frac{8}{2} = 4cm$$

5.2

$$AE = 12cm$$

$$\frac{ED}{AE} = \frac{1}{2} \quad [BE\ //\ DC;\ prop\ \ theorem]$$

$$\frac{ED}{12} = \frac{1}{2}$$

$$ED = 6cm$$

QUESTION 7

7.1

$\hat{T}_3 = 90^\circ$ [∠s in semi circle]

$X\hat{B}Y = 90^\circ$ [given]

∴ $\hat{T}_3 = X\hat{B}Y$ [both = 90°]

BX // TP [corresp ∠s =]

7.2

In ΔXBT and ΔXBY

$X\hat{B}T = X\hat{B}Y$ [common ∠]

$\hat{X}_2 = \hat{T}_2$ [alt ∠s; BX // TP]

$\hat{T}_2 = \hat{Y}$ [tan chord theorem]

$\hat{X}_2 = \hat{Y}$ [both = $\hat{T}_2$]

$\hat{T}_1 = B\hat{X}Y$ [3^{rd} ∠ of the Δ]

∴ ΔXBT /// ΔXBY [∠∠∠]

$$\frac{XB}{YB} = \frac{XT}{YX} \quad [///\ \Delta s]$$

QUESTION 8

8.1.1	$B\hat{A}E = 90°$	$\angle$ semi circle	✓S ✓R	(2)
8.1.2	$\hat{E}_1 = 80°$	opp angles cyclic quad	✓S ✓R	(2)
8.1.3	$\hat{D}_1 = 45°$	ext $\angle$ of ΔFED	✓S ✓R	
8.2	$\hat{B}_1 = 35°$	*Interior* $\angle$ of Δ	✓S ✓R	(4)
	$\hat{F} = 35°$	*given*	✓S	
	$\therefore AB \parallel CF$	*Alternate* $\angle s =$	✓R	

[10]

QUESTION 9

9.1 $DO.OD = OD(OD + OA)$ ✓ $OD(OD+OA)$

$= OD^2 + OD.OA$ (1)

9.2 In ΔDAB and ΔDCO

$\hat{D} = \hat{D}$ (common) ✓S $\hat{D} = \hat{D}$

$\hat{C}_2 = 90°$ (line from centre to midpt of a chord/Midpt thm) ✓S $\hat{C}_2 = 90°$ ✓R

$\hat{C}_2 = \hat{A}$

$\hat{B} = \hat{O}_3$ $(3^{rd} \angle \text{ of a } \Delta)$

$\therefore \Delta DAB \parallel\mid \Delta DCO$ $(\angle\angle\angle)$ ✓S $\hat{C}_2 = \hat{A}$

$\therefore \frac{DA}{DC} = \frac{AB}{CO} = \frac{DB}{DO}$ ✓S $\hat{B} = \hat{O}_3$

$DA.DO = DC.DB$ ✓ $\frac{DA}{DC} = \frac{AB}{CO} = \frac{DB}{DO}$

$OD^2 + OD.OA = DC.2DC$ ✓

$OD^2 + OD.OA = 2DC^2$ $OD^2 + OD.OA = 2DC^2$ (7)

[8]

QUESTION 10

	$P\hat{T}R = 90°$	$(\angle in\ semi-circle)$	✓S/R	
10	$x = 90^0 + \hat{R}$	$(\text{ext} \angle \text{of } \Delta)$	✓S/R	
	$\therefore \hat{R} = x - 90°$			
	$S\hat{T}P = x - 90°$	$(\text{tan chord theorem})$	✓S ✓R	
	$x + x - 90° + y = 180°$	$(\text{sum of } \angle \text{s in } \Delta)$	✓S	
	$\therefore y = 270° - 2x$		✓answer	(6)

QUESTION 11

11.1.1 In ΔOBC

$\hat{B}_2 = \hat{C}_3$ $(\angle \text{s opposite} = \text{radii})$ ✓S ✓R

$\hat{B}_2 = 90° - 2x$ $(\text{sum of } \angle\text{'s of a } \Delta)$ ✓ $\hat{B}_2 = 90° - 2x$

(2)

11.1.2 $\hat{A}_3 = 2x$ ✓S ✓R

$(\angle \text{at centre} = 2 \times \angle \text{at circumference})$

$\hat{A}_3 = \hat{C}_1 + \hat{E}$ $(\text{ext} \angle \text{of } \Delta)$ ✓S

But $AB = AC = AE$ (given) ✓S

$\hat{C}_1 = \hat{E}$ $(\angle \text{s opposite} = \text{sides})$

$\therefore \hat{E} = x$ ✓ $\hat{E} = x$

(5)

11.1.3 $\hat{B}_1 + \hat{B}_2 = \hat{C}_2 + \hat{C}_3$ $(\angle \text{s opposite} = \text{sides})$ ✓S

$\hat{B}_1 = \hat{C}_2 = 180° - (2x + 90° - 2x + 90° - 2x)$ ✓S

$(\text{sum of } \angle\text{'s of a } \Delta)$

$\therefore \hat{C}_2 = x$ ✓S

(3)

11.2 $\hat{A}_1 = \hat{C}$ (ext. $\angle$s of a cyclic quadrilateral) ✓S ✓R

$\hat{A}_1 = 90° - 2x + x + x$

$\hat{A}_1 = 90°$ ✓ $\hat{A}_1 = 90°$

ED is a diameter of circle (line subtends $90°$ $\angle$)/ ✓R

(converse of $\angle$ in a semi – circle) (4)

QUESTION 12

12.1 tangent-chord theorem (1)

12.1.2 In ΔABC and ΔADB:

$\hat{A}_1 = \hat{A}_1$ (common) ✓S

$\hat{B}_1 = \hat{D}_1$ (proved in 10.2.1) ✓S

$\therefore \Delta ABC \,|||\, \Delta ADB$ ($\angle\angle\angle$) ✓R

OR

$\hat{A}_1 = \hat{A}_1$ (common)

$\hat{B}_1 = \hat{D}_1$ (proved in 10.2.1) ✓S

$B\hat{C}A = \hat{B}_2$ ($\angle$s of a $\Delta = 180°$) ✓S

$\therefore \Delta ABC \,|||\, \Delta ADB$ ✓R

(3)

12.1.3 $\hat{E}_2 = \hat{F}_1$ (alternate $\angle$s; EA || GF) ✓S✓R

$\hat{F}_1 = \hat{D}_2$ (ext. $\angle$s of a cyclic quad DGFC) ✓S✓R

$\therefore \hat{E}_2 = \hat{D}_2$ (4)

12.1.4 In ΔAEC and ΔADE:

$\hat{A}_2 = \hat{A}_2$ (common) ✓S

$\hat{E}_2 = \hat{D}_2$ (proved in 10.2.3) ✓S

$\therefore \Delta AEC \mathbin{|||} \Delta ADE$ (∠∠∠) ✓R

$\therefore \frac{AE}{AD} = \frac{AC}{AE}$ ✓S

$\therefore AE^2 = AD \times AC$

OR

In ΔAEC and ΔADE:

$\hat{A}_2 = \hat{A}_2$ (common)

$\hat{E}_2 = \hat{D}_2$ (proved in 10.2.3) ✓S

$A\hat{C}E = \hat{G}_1$ (∠s of a Δ = 180° OR ext ∠ of cyclic quad DGFE) ✓S

$\therefore \Delta AEC \mathbin{|||} \Delta ADE$ ✓R

$\therefore \frac{AE}{AD} = \frac{AC}{AE}$ ✓S

$\therefore AE^2 = AD \times AC$

(4)

12.1.5 $\frac{AB}{AD} = \frac{AC}{AB}$ $(\Delta ABC \mathbin{|||} \Delta ADB)$ ✓S

$AB^2 = AD \times AC$

$= AE^2$ (from 10.2.4) ✓S

$\therefore AB = AE$ ✓S

(3)

[16]

QUESTION 13

13.1

$\hat{N}_1 = \hat{C}_4$	(tan chord theorem)	✓S/R
$\hat{N}_1 = \hat{K}_2$	(tan chord theorem)	✓S/R
$\therefore \hat{C}_4 = \hat{K}_2$		
$CG \parallel KM$	(corresp $\angle$s =)	
$\dfrac{KC}{KN} = \dfrac{MG}{MN}$	(line $\parallel$ to one side of Δ OR prop theorem)	✓S/R ✓R
		(4)

13.2

$\hat{C}_4 = \hat{K}_2$	(proved)	
$\hat{C}_4 = \hat{G}_2$	($\angle$s opposite = sides)	✓ $\hat{C}_4 = \hat{G}_2$ ✓R
$\therefore \hat{G}_2 = \hat{K}_2$		
$\therefore$ KMGC is a cyclic quad	(ext $\angle$ = int opp $\angle$)	✓R
		(3)

13.3 In ΔMCG and ΔMNC:

$\hat{M}_2 = \hat{M}_2$	(common)	✓S
$\hat{C}_3 = \hat{N}_2$	(tan chord theorem)	✓S/R
$\hat{G}_1 = \hat{C}_3 + \hat{C}_4$	(sum of $\angle$s in Δ)	
$\therefore \Delta MCG \parallel\!\mid \Delta MNC$	($\angle\angle\angle$)	✓R
		(4)

13.4 $\frac{MC}{MG}=\frac{MN}{MC}$ $(\Delta^s\ |||)$ ✓S/R

$MC^2 = MG.MN$

$\frac{MC^2}{MN^2}=\frac{MG.MN}{MN^2}$ ✓S

$=\frac{MG}{MN}$

$\frac{KC}{KN}=\frac{MG}{MN}$ (proved) ✓S

$\frac{MC^2}{MN^2}=\frac{KC}{KN}$ ✓S

(4)

[19]

QUESTION 14

14.1 alt $\angle$s, YT || RQ ✓R

14.2.1 $\frac{BP}{BE}=\frac{BS}{BF}$ (Prop theorem, EF || PS) ✓S✓R

$BE^2=\frac{BP.BF}{BS}$ **(2)**

14.1.2 In ΔBGP and ΔBEG:

1) $\hat{G}_1=\hat{P}_1$ (tan chord theorem) ✓S✓R

2) $\hat{B}=\hat{B}$ (common)

$\therefore \Delta BGP\ |||\ \Delta BEG$ $(\angle\angle\angle)$ ✓S/R

OR ✓S/R

In ΔBGP and ΔBEG:

1) $\hat{G}_1 = \hat{P}_1$ (tan chord theorem) ✓S✓R
2) $\hat{B} = \hat{B}$ (common)
3) $B\hat{G}P = B\hat{E}G$ (sum of $\angle$s in Δ) ✓S/R

$\therefore \Delta BGP \;|||\; \Delta BEG$

✓S
(4)

14.1.3 $\frac{BG}{BE} = \frac{BP}{BG}$ $\quad \Delta BGP \;|||\; \Delta BEG$

$\therefore BG^2 = BP.BE$ ✓S

$BG^2 = BP.\frac{BP.BF}{BS}$ ✓S

$BG^2 = \frac{BP^2.BF}{BS}$ ✓Subst

$\therefore \frac{BG^2}{BP^2} = \frac{BF}{BS}$

(4)

[10]

QUESTION 15

15.1.1 $\frac{FC}{20} = \frac{4}{5}$ $\quad (EF \parallel AD)$ ✓S ✓R

$\therefore FC = 16$ ✓answer
(3)

15.1.2 $\frac{36}{DB} = \frac{4}{5}$ $\quad (DE \parallel AB)$ ✓ $DC = 16$

✓S ✓R

$\therefore DB = 45$

✓answer
(4)

15.2

$$\frac{\text{Area of } \Delta \text{ECF}}{\text{Area of } \Delta \text{ABC}} = \frac{\frac{1}{2}.4k.8.\sin C}{\frac{1}{2}.9k.81.\sin C}$$ ✓ $\frac{1}{2}.4k.8.\sin C$

✓ $\frac{1}{2}.9k.40.5.\sin C$

$$\frac{\text{Area of } \Delta \text{ECF}}{\text{Area of } \Delta \text{ABC}} = \frac{32}{81}$$ ✓✓answer

QUESTION 16

16.1.1 Angles in a semi-circle ✓R
(1)

16.1.2 Exterior $\angle$ of a quad = opp interior $\angle$

OR

Opp $\angle$s of a quad supplementary ✓R
(1)

16.1.3 tangent chord theorem ✓R
(1)

16.2.1 In Δ AEC

$\hat{\text{E}} = 180° - (90° + \text{x})$ (sum of $\angle$s in Δ)

$\hat{\text{E}} = 90° - x$ ✓S

$\hat{\text{D}}_1 = 180° - (90° + \text{x})$ ($\angle$s on a straight line) ✓S

$\hat{\text{D}}_1 = 90° - x$

$\therefore$ AD = AE (sides opp = $\angle$s) ✓R
(3)

16.2.2 In $\Delta\,ADB$ and $\Delta\,ACD$ ✓S

$\hat{A}_2 = \hat{A}_2$ (common)

$\hat{D}_2 = C$ (proven) ✓S

$\hat{B}_2 = \hat{D}_2 + \hat{D}_3$ (sum of $\angle$s in Δ) ✓S

$\therefore\ \Delta\,ADB\ |||\ \Delta\,ACD$

OR

In $\Delta\,ADB$ and $\Delta\,ACD$

$\hat{A}_2 = \hat{A}_2$ (common) ✓S

$\hat{D}_2 = C$ (proven) ✓S

$\therefore\ \Delta\,ADB\ |||\ \Delta\,ACD$ ($\angle\angle\angle$) ✓R

(3)

16.3.1 $\dfrac{AD}{AC} = \dfrac{AB}{AD}$ ($|||\,\Delta$s) ✓ratio

$AD^2 = AC.AB$

$= 3r.r$ ✓substitution

$= 3r^2$

(2)

16.3.2 $AD = AE = \sqrt{3}\,r$ $\quad$ (from 11.2.2(a)) & 11.2.3(a)

$AB = r$ and $BC = 2r$ $\therefore AC = 3r$ $\quad$ ✓AC ito r

In ΔACE:

$\tan \hat{E} = \dfrac{AC}{AE}$ $\quad$ ✓trig ratio

$= \dfrac{3r}{\sqrt{3}\,r} = \sqrt{3}$ $\quad$ ✓simplification

$\therefore \hat{E} = 60°$

$\therefore \hat{D}_1 = 60°$ $\quad$ (from 11.2.2(a))

$\therefore \hat{A}_1 = 60°$ $\quad$ ($\angle$s of $\Delta = 180°$) $\quad$ ✓all 3 $\angle s = 60°$

$\therefore$ Δ ADE is a cyclic quad

OR

$\dfrac{AD}{AC} = \dfrac{DB}{CD}$ $\quad$ ($|||\Delta s$)

$\dfrac{\sqrt{3}\,r}{3r} = \dfrac{DB}{CD}$ $\quad$ ✓ $\dfrac{\sqrt{3}\,r}{3r} = \dfrac{DB}{CD}$

$\tan x = \dfrac{1}{\sqrt{3}}$ $\quad$ ✓ $\tan x = \dfrac{1}{\sqrt{3}}$

$\therefore$ In $\Delta BDC : x = 30°$ $\quad$ ✓ $x = 30°$

$\therefore \hat{E} = 60°$

$\therefore \hat{D}_1 = 60°$ $\quad$ (from 11.2.2(a))

$\therefore \hat{A}_1 = 60°$ $\quad$ ($\angle$s of $\Delta = 180°$) $\quad$ ✓all 3 $\angle s = 60°$

$\therefore$ Δ ADE is a cyclic quad

OR

$$\frac{AD}{AC} = \frac{DB}{CD} \qquad (|||\Delta s)$$

$$\frac{\sqrt{3}\,r}{3r} = \frac{DB}{CD} \quad \therefore\ BD = \frac{CD}{\sqrt{3}}$$

$$DC^2 = BC^2 - DB^2$$

$$DC^2 = 4r^2 - \frac{CD^2}{3}$$

$$3DC^2 = 12r^2 - CD^2$$

✓ $BD = \frac{CD}{\sqrt{3}}$

$$4DC^2 = 12r^2$$

$$DC = \sqrt{3}r$$

$$EC^2 = EA^2 + AC^2$$

$$= 3r^2 + 9r^2$$

✓ $DC = \sqrt{3}r$

$$EC = 2\sqrt{3}r$$

$$\therefore ED = EC - DC$$

$$= \sqrt{3}r$$

✓ $EC = 2\sqrt{3}r$

$$\therefore ED = EA = AD$$

$\therefore ADE$ is equilateral

✓ $ED = EA = AD$

ACCEPTABLE REASONS: EUCLIDEAN GEOMETRY

In order to have some kind of uniformity, the use of the following shortened versions of the theorem statements is encouraged.

Acceptable reasons: Euclidean Geometry (English)

THEOREM STATEMENT	ACCEPTABLE REASON(S)
LINES	
The adjacent angles on a straight line are supplementary.	**∠s on a str line**
If the adjacent angles are supplementary, the outer arms of these angles form a straight line.	**adj ∠s supp**
The adjacent angles in a revolution add up to 360°..	**∠s round a pt OR ∠s in a rev**
Vertically opposite angles are equal.	**vert opp ∠s =**
If AB || CD, then the alternate angles are equal.	**alt ∠s; AB || CD**
If AB || CD, then the corresponding angles are equal.	**corresp ∠s; AB || CD**
If AB || CD, then the co-interior angles are supplementary.	**co-int ∠s; AB || CD**
If the alternate angles between two lines are equal, then the lines are parallel.	**alt ∠s =**
If the corresponding angles between two lines are equal, then the lines are parallel.	**corresp ∠s =**
If the co-interior angles between two lines are supplementary, then the lines are parallel.	**coint ∠s supp**
TRIANGLES	
The interior angles of a triangle are supplementary.	**∠ sum in Δ OR sum of ∠s in Δ OR Int ∠s Δ**
The exterior angle of a triangle is equal to the sum of the interior opposite angles.	**ext ∠ of Δ**
The angles opposite the equal sides in an isosceles triangle are equal.	**∠s opp equal sides**
The sides opposite the equal angles in an isosceles triangle are equal.	**sides opp equal ∠s**
In a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.	**Pythagoras OR Theorem of Pythagoras**
If the square of the longest side in a triangle is equal to the sum of the squares of the other two sides then the triangle is right-angled.	**Converse Pythagoras OR Converse Theorem of Pythagoras**
If three sides of one triangle are respectively equal to three sides of another triangle, the triangles are congruent.	**SSS**
If two sides and an included angle of one triangle are respectively equal to two sides and an included angle of another triangle, the triangles are congruent.	**SAS OR S∠S**

THEOREM STATEMENT	ACCEPTABLE REASON(S)
If two angles and one side of one triangle are respectively equal to two angles and the corresponding side in another triangle, the triangles are congruent.	**AAS OR ∠∠S**
If in two right-angled triangles, the hypotenuse and one side of one triangle are respectively equal to the hypotenuse and one side of the other, the triangles are congruent	**RHS OR 90°HS**
The line segment joining the midpoints of two sides of a triangle is parallel to the third side and equal to half the length of the third side	**Midpt Theorem**
The line drawn from the midpoint of one side of a triangle, parallel to another side, bisects the third side.	**line through midpt || to 2nd side**
A line drawn parallel to one side of a triangle divides the other two sides proportionally.	**line || one side of Δ** **OR prop theorem; name || lines**
If a line divides two sides of a triangle in the same proportion, then the line is parallel to the third side.	**line divides two sides of Δ in prop**
If two triangles are equiangular, then the corresponding sides are in proportion (and consequently the triangles are similar).	**||| Δs OR equiangular Δs**
If the corresponding sides of two triangles are proportional, then the triangles are equiangular (and consequently the triangles are similar).	**Sides of Δ in prop**
If triangles (or parallelograms) are on the same base (or on bases of equal length) and between the same parallel lines, then the triangles (or parallelograms) have equal areas.	**same base; same height OR** **equal bases; equal height**
CIRCLES	
The tangent to a circle is perpendicular to the radius/diameter of the circle at the point of contact.	**tan ⊥** **radius tan** **⊥ diameter**
If a line is drawn perpendicular to a radius/diameter at the point where the radius/diameter meets the circle, then the line is a tangent to the circle.	**line ⊥ radius OR** **converse tan ⊥ radius** **OR converse tan ⊥** **diameter**
The line drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord.	**line from centre to midpt of chord**
The line drawn from the centre of a circle perpendicular to a chord bisects the chord.	**line from centre ⊥ to chord**
The perpendicular bisector of a chord passes through the centre of the circle;	**perp bisector of chord**
The angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre)	**∠ at centre = 2 ×∠ at circumference**
The angle subtended by the diameter at the circumference of the circle is 90°.	**∠s in semi- circle OR** **diameter subtends right angle**
If the angle subtended by a chord at the circumference of the circle is 90°. then the chord is a diameter.	**chord subtends 90° OR** **converse ∠s in semi -circle**

THEOREM STATEMENT	ACCEPTABLE REASON(S)
Angles subtended by a chord of the circle, on the same side of the chord, are equal	**$\angle$s in the same seg.**
If a line segment joining two points subtends equal angles at two points on the same side of the line segment, then the four points are concyclic.	**line subtends equal $\angle$s OR** **converse $\angle$s in the same seg.**
Equal chords subtend equal angles at the circumference of the circle.	**equal chords; equal $\angle$s**
Equal chords subtend equal angles at the centre of the circle.	**equal chords; equal $\angle$s**
Equal chords in equal circles subtend equal angles at the circumference of the circles.	**equal circles; equal chords; equal $\angle$s**
Equal chords in equal circles subtend equal angles at the centre of the circles.	**equal circles; equal chords; equal $\angle$s**
The opposite angles of a cyclic quadrilateral are supplementary	**opp $\angle$s of cyclic quad**
If the opposite angles of a quadrilateral are supplementary then the quadrilateral is cyclic.	**opp $\angle$s quad supp OR** **converse opp $\angle$s of cyclic quad**
The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.	**ext $\angle$ of cyclic quad**
If the exterior angle of a quadrilateral is equal to the interior opposite angle of the quadrilateral, then the quadrilateral is cyclic.	**ext $\angle$ = int opp $\angle$ OR** **converse ext $\angle$ of cyclic quad**
Two tangents drawn to a circle from the same point outside the circle are equal in length	**Tans from common pt OR** **Tans from same pt**
The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.	**tan chord theorem**
If a line is drawn through the end-point of a chord, making with the chord an angle equal to an angle in the alternate segment, then the line is a tangent to the circle.	**converse tan chord theorem OR** **$\angle$ between line and chord**
QUADRILATERALS	
The interior angles of a quadrilateral add up to 360.	**sum of $\angle$s in quad**
The opposite sides of a parallelogram are parallel.	**opp sides of ||m**
If the opposite sides of a quadrilateral are parallel, then the quadrilateral is a parallelogram.	**opp sides of quad are ||**
The opposite sides of a parallelogram are equal in length.	**opp sides of ||m**
If the opposite sides of a quadrilateral are equal , then the quadrilateral is a parallelogram.	**opp sides of quad are =** **OR converse opp sides of a parm**
The opposite angles of a parallelogram are equal.	**opp $\angle$s of ||m**
If the opposite angles of a quadrilateral are equal then the quadrilateral is a parallelogram.	**opp $\angle$s of quad are = OR** **converse opp angles of a parm**
The diagonals of a parallelogram bisect each other.	**diag of ||m**
If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.	**diags of quad bisect each other** **OR converse diags of a parm**
If one pair of opposite sides of a quadrilateral are equal and parallel, then the quadrilateral is a parallelogram.	**pair of opp sides = and ||**
The diagonals of a parallelogram bisect its area.	**diag bisect area of ||m**
The diagonals of a rhombus bisect at right angles.	**diags of rhombus**

The diagonals of a rhombus bisect the interior angles.	**diags of rhombus**
All four sides of a rhombus are equal in length.	**sides of rhombus**
All four sides of a square are equal in length.	**sides of square**
The diagonals of a rectangle are equal in length.	**diags of rect**
The diagonals of a kite intersect at right-angles.	**diags of kite**
A diagonal of a kite bisects the other diagonal.	**diag of kite**
A diagonal of a kite bisects the opposite angles	**diag of kite**

TERMINOLOGY

Term	Explanation
Euclidean Geometry	Geometry based on the postulates of Euclid. Euclidean geometry deals with space and shape using a system of logical deductions
theorem	A statement that has been proved based on previously established statements
converse	A statement formed by interchanging what is given in a theorem and what is to be proved
rider	A problem of more than usual difficulty added to another on an examination paper
radius	Straight line from the centre to the circumference of a circle or sphere. It is half of the circle's diameter
diameter	Straight line going through the centre of a circle connecting two points on the circumference
chord	Line segment connecting two points on a curve. When the chord passes through the centre of a circle it is called the diameter
quadrilateral	A 4-sided closed shape (polygon)
cyclic quadrilateral	A quadrilateral whose vertices all lie on a single circle. This circle is called the circumcircle or circumscribed circle, and the vertices are said to be concyclic
diagonal	A straight line joining two opposite vertices (corners) of a straight sided shape. It goes from one corner to another but is not an edge
circumference	The distance around the edge of a circle (or any curved shape). It is a type of perimeter
segment	The area bound by a chord and an arc
arc	Part of the circumference of a circle
sector	The area bound by two radii and an arc
Corollary (Theorem that follows on from another theorem)	A statement that follows with little or no proof required from an already proven statement. For example, it is a theorem in geometry that the angles opposite two congruent sides of a triangle are also congruent (isosceles triangle). A corollary to that statement is that an equilateral triangle is also equiangular.
Theorem of Pythagoras	In any right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.
hypotenuse	The longest side in a right-angled triangle. It is opposite the right angle.

Complementary angles	Angles that add up to 90°.
Supplementary angles	Angles that add up to 180°.
Vertically opposite angles	Non-adjacent opposite angles formed by intersecting lines.
Intersecting lines	Lines that cross each other.
Perpendicular lines	Lines that intersect each other at a right angle.
parallel lines	Lines the same distance apart at all points. Two or more lines are parallel if they have the same slope (gradient).
transversal	A line that cuts across a set of lines (usually parallel).
Corresponding angles	Angles that sit in the same position on each of the parallel lines in the position where the transversal crosses each line.
alternate angles	Angles that lie on different parallel lines and on opposite sides of the transversal.
co-interior angles	Angles that lie on different parallel lines and on the same side of the transversal.
congruent	The same. Identical.
similar	Looks the same. Equal angles and sides in proportion.
proportion	A part, share, or number considered in comparative relation to a whole. The equality of two ratios. An equation that can be solved.
ratio	The comparison of sizes of two quantities of the same unit. An expression.
area	The space taken up by a two-dimensional polygon.
tangent	Line that intersects with a circle at only one point (the point of tangency)
Point of tangency	The point of intersection between a circle and its tangent line
exterior angle	The angle between any side of a shape, and a line extended from the next side
subtend	The angle made by a line or arc
polygon	A closed 2D shape in which all the sides are made up of line segments. A polygon is given a name depending on the number of sides it has. A circle is not a polygon as although it is a closed 2D shape it is not made up of line segments
Radii (plural of radius)	This is common when triangles are drawn inside circles – look out for lines drawn from the centre. Remember that all radii are equal in length in a circle

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